Appendix to Notes 1

Cardinality

Assume that A and B are two finite sets and let $|A|$ and $|B|$ denote the number of elements in the sets.

Note that $|A| \leq |B|$ if, and only if, there exists a one-to-one function from A to B.

Alternatively, $|A| \leq |B|$ if, and only if, there exists an onto function from B to A.

Assume now that we are given two sets C and D with a bijection between them. A bijection is one-to-one from C to D (so $|C| \leq |D|$) and onto from C to D (so $|C| \geq |D|$). Hence $|C| = |D|$.

We turn this around in the following definition.

Definition Two sets (including infinite sets) have the same *cardinality* if there exists a bijection between them.

Example 1 For a finite set E we say that E has cardinality n, and write $|E| = n$ if there exists a bijection from E to $\{1, 2, 3, ..., n\}$. If the bijection is $g : E \to \{1, 2, 3, ..., n\}$ then we can write $E = \{e_1, e_2, ..., e_n\}$ where $g(e_k) = k$ for all $1 \leq k \leq n$.

Definition If for a set E there exists a bijection between E and $\mathbb{N}$ we say that E is *countable*. (Though it is not standard we will also say that finite sets are countable.) For an infinite countable set we write $|E| = \aleph_0$. Again, if the bijection is $h : E \to \mathbb{N}$ then we can *enumerate,* or *list the elements of,* E as $E = \{e_1, e_2, e_3, ...\}$ where $h(e_k) = k$ for all $k \geq 1$.

Example 2 The set of integers, $\mathbb{Z}$, is countable. For a possible bijection take

$$h : \mathbb{Z} \to \mathbb{N},\ n \mapsto \begin{cases} 2n - 1 & \text{if } n \geq 1 \\ 2 - 2n & \text{if } n \leq 0. \end{cases}$$

This would give an enumerate of $\mathbb{Z}$ as $\{1, 0, 2, -1, 3, -2, 4, -3, ...\}$.

Example 3 The Cartesian product, $\mathbb{N} \times \mathbb{N}$ is countable. List the ordered pairs in the following array.

$$\begin{array}{lllll} (1,1)^{(1)} & (1,2)^{(2)} & (1,3)^{(6)} & (1,4)^{(7)} & ...^{(15)} \\ (2,1)^{(3)} & (2,2)^{(5)} & (2,3)^{(8)} & ...^{(14)} & \\ (3,1)^{(4)} & (3,2)^{(9)} & (3,3)^{(13)} & ... & \\ (4,1)^{(10)} & (4,2)^{(12)} & ... & & \\ ...^{(11)} & ... & & & \end{array}$$

The superscripts here indicate how to map the elements of $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

Example 4 The Cartesian product, $\mathbb{Z} \times \mathbb{Z}$ is countable. We know from Example 2 that we can enumerate $\mathbb{Z}$ as $\{r_1, r_2, r_3, ...\}$ say, so we can list $\mathbb{Z} \times \mathbb{Z}$ as follows.

$$\begin{array}{lllll}
(r_1, r_1)^{(1)} & (r_1, r_2)^{(2)} & (r_1, r_3)^{(6)} & (r_1, r_4)^{(7)} & ...^{(15)} \\
(r_2, r_1)^{(3)} & (r_2, r_2)^{(5)} & (r_2, r_3)^{(8)} & ...^{(14)} & \\
(r_3, r_1)^{(4)} & (r_3, r_2)^{(9)} & (r_3, r_3)^{(13)} & ... & \\
(r_4, r_1)^{(10)} & (r_4, r_2)^{(12)} & ... & & \\
...^{(11)} & ... & & &
\end{array}$$

The same mapping as in Example 3 suffices.

Example 5 Similarly, $\mathbb{Z}^n$ is countable for any $n \geq 1$.

Theorem 1 *If A is countable and $B \subseteq A$ then B is countable.*

Proof If B is finite there is nothing to prove.

Assume that B is infinite.

Let $f : \mathbb{N} \to A$ be a bijection that exists since A is countable and enumerate the distinct elements of A as $\{a_1, a_2, a_3, ...\}$. We look at those elements that lie in B. Because of the ordering on A we get an ordering on B and we can relabel so $B = \{b_1, b_2, b_3, ...\}$ where b_1 is the first element of A in B, i.e.

$$b_1 = a_{n_1} \text{ where } n_1 = \min\{n : a_n \in B\},$$

b_2 is second element of A in B, i.e

$$b_2 = a_{n_2} \text{ where } n_2 = \min\{n : a_n \in B \setminus \{b_1\}\},$$

and in general

$$b_{r+1} = a_{n_{r+1}} \text{ where } n_{r+1} = \min\{n : a_n \in B \setminus \{b_1, b_2, ..., b_r\}\}.$$

Note that the b_i are distinct and every element of B occurs in the list $b_1, b_2, b_3.....$

To see this last statement take any element b of B. The since $B \subseteq A$ we have $b = a_m$ for some m. The list $n_1 < n_2 < n_3....$ is infinite and so at some point we must have $n_t \leq m < n_{t+1}$. If we had $n_t < m < n_{t+1}$ then we would have the existance of $a_m = b \in B$ with

$$\min\{n : a_n \in B \setminus \{b_1, b_2, ..., b_t\}\} > m > \min\{n : a_n \in B \setminus \{b_1, b_2, ..., b_{t-1}\}\}.$$

The first inequality tells us that $b \in B \setminus \{b_1, b_2, ..., b_t\}$ where as the second tells us that $b \notin B \setminus \{b_1, b_2, ..., b_{t-1}\}$, which is contradictory. Hence we must have $m = n_t$ so the b must occur in our listing as b_t.

Since every element of B occurs in the list $b_1, b_2, b_3....$ we have that the map $g : \mathbb{N} \to B, n \mapsto b_n$ is onto. We can see that it is 1-1 by assuming $g(s) = g(t)$ so $b_s = b_t$, i.e $a_{n_s} = a_{n_t}$. But then $n_s = n_t$ since the labeled elements of A are distinct. Similarly the n_j are distinct so we must have $s = t$. Hence g is 1-1, thus it is a bijection. Hence B is countable. ■

Example 6 $\mathbb{Q}$ is countable.

Verification Write each element as r/s with $s \in \mathbb{N}, r \in \mathbb{Z}$ and s, r coprime (so the fraction is in *lowest terms*.) Then we can map $\mathbb{Q}$ into a subset of the array in Example 4 by $r/s \mapsto (r, s)$. The image of this map is a subset of an array that we know is countable and so is countable, and the map is a bijection, so $\mathbb{Q}$ is countable.

Example 7 A countable union of countable sets is countable.

Verification A countable collection of sets means that they can be listed as $S_1, S_2, S_3, ...$, say. Each S_i is countable and so they, in turn, can be listed as $S_i = \{a_{i1}, a_{i2}, a_{i3}, ...\}$. Then $\bigcup_{i=1}^{\infty} S_i$ is **contained** in the following array. (I use the word contained since the array may contain repeated elements that are counted only once in the union.)

$$\begin{array}{lllll}
a_{11}\ ^{(1)} & a_{12}\ ^{(2)} & a_{13}\ ^{(6)} & a_{14}\ ^{(7)} & ....\ ^{(15)} \\
a_{21}\ ^{(3)} & a_{22}\ ^{(5)} & a_{23}\ ^{(8)} & ....\ ^{(14)} & \\
a_{31}\ ^{(4)} & a_{32}\ ^{(9)} & a_{33}\ ^{(13)} & \ldots & \\
a_{41}\ ^{(10)} & a_{42}\ ^{(12)} & \ldots & & \\
\ldots^{(11)} & \ldots & & &
\end{array}$$

As in previous examples this array is countable and so $\bigcup_{i=1}^{\infty} S_i$ is countable.

Definition A (real) *algebraic number* if a root of any polynomial of the form

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_2 x^2 + a_1 x + a_0, \tag{1}$$

for any $n \geq 1$ and where the $a_i \in \mathbb{Z}$ for all i. For example $\sqrt{2}$ is algebraic.

If an algebraic number α is a root of a polynomial of degree n but no polynomial of smaller degree we say that α *has degree* n.

Example 8 The set of algebraic numbers is countable.

Verification For each $m \geq 1$ define $R_m \subseteq \mathbb{R}$ to be the set of real roots of all polynomials as in (1) but with degree equal to m. For each polynomial there are at most m roots and there are at most $\mathbb{Z}^{m+1}$ possible polynomials. So R_m is a countable union of finite sets and therefore countable. Hence the set of algebraic numbers, $\bigcup_{n=1}^{\infty} R_n$, is a countable union of countable sets, hence countable.

Example 9 The set of real numbers, $\mathbb{R}$, is uncountable.

Verification It suffices to show that $[0, 1)$ is uncountable.

Assume that $[0, 1)$ is countable and so list the elements in non-terminating decimal form. The list will start as

$$\begin{aligned} a_1 &= 0.a_{11}a_{12}a_{13}a_{14}... \\ a_2 &= 0.a_{21}a_{22}a_{23}a_{24}... \\ a_3 &= 0.a_{31}a_{32}a_{33}a_{34}... \\ a_4 &= 0.a_{41}a_{42}a_{43}a_{44}... \\ &\vdots \end{aligned}$$

Consider now the number $b = 0.b_1b_2b_3b_4... \in [0, 1)$ defined by

$$b_j = \begin{cases} a_{jj} + 1 & \text{if } 0 \leq a_{jj} \leq 8 \\ 1 & \text{if } a_{jj} = 9. \end{cases}$$

Obviously, for every $j \geq 1$ the number b differs from the j^{th} element in the list in the j^{th} decimal place. Hence b cannot occur in the list. This contradicts the assertion that the list contains all numbers in $[0, 1)$. Thus the assumption is false, and $\mathbb{R}$ is not countable. This is known as Cantor's diagonal argument. ■

Definition If a set, A, has the same cardinality as $\mathbb{R}$ we write $|A| = c$.

Example 10 We say that the non-algebraic real numbers are called *transcendental*. Combining examples 8 and 9 we see that the set of transcendental numbers is uncountable. So in some sense there are far more transcendental numbers than algebraic numbers but it is far harder to recognize that a given number is transcendental. For instance it is only relatively recently that π and e have been proved to be transcendental. (It is hard enough to show they are irrational!)

Note If A is a finite set then the number of subsets is given by $2^{|A|}$. We use this observation to assign a symbol to the cardinality of the collection of subsets of an infinite set.

Definition If A is a set (possibly infinite) the collection of all subsets of A (i.e. the power set of A) will be denoted by 2^A (though we still use the $P(A)$ notation) and the cardinality of 2^A will be denoted by $2^{|A|}$.

Example 11 The set $2^{\mathbb{N}}$ is uncountable.

The proof of this is similar to the proof that $\mathbb{R}$ is uncountable. So suppose $2^{\mathbb{N}}$ is countable. Let $v_1, v_2, v_3, ...$ be some enumeration of the subsets of $\mathbb{N}$. Define a new set v by saying k is in v if, and only if, k is not in v_k. Then, for every $k \geq 1$, one and only one of the pair v and v_k contains k, and so v differs from v_k. Hence v does not appear in the enumeration, contradicting our assumption.

Definition We say that a set A has *greater cardinality* than a set B if A cannot be put in one-to-one correspondence with B, but a proper subset of A can be put in one-to-one correspondence with B.

So the cardinality of $2^{\mathbb{N}}$ is greater than $\mathbb{N}$. The question arises whether there exists a set A with greater cardinality than $\mathbb{N}$ and lesser cardinality than $2^{\mathbb{N}}$. It was asserted by Cantor in the *Continuum Hypothesis* that no such set existed. In particular, this means that every infinite subset of $2^{\mathbb{N}}$ is in one-to-one correspondence with either $\mathbb{N}$ or $2^{\mathbb{N}}$. Alternatively, every uncountable set, A, satisfies $|A| \geq c$. Strangely, it is not expected that we will ever know if this hypothesis is true or not.

Example 12 It is not too hard to show that $2^{\aleph_0} = c$. (Suggestion, for every $x \in [0,1]$, written in base 2 as a non-terminating expansion, $x = 0.x_1x_2x_3x_4....$, define a subset V of $\mathbb{N}$ by $k \in V$ if, and only if, $x_k = 1$.)

Topological Space results

Theorem 1.3 (Heine-Borel) *If $[a,b] \subseteq \mathbb{R}$ is covered by a collection of open intervals, so $[a,b] \subseteq \bigcup_{i\in I}(c_i,d_i)$, then there exists a finite sub-collection of the (c_i,d_i), which can relabeled as $1 \leq i \leq N$ such that $[a,b] \subseteq \bigcup_{i=1}^{N}(c_i,d_i)$.*

Proof We have

$$[a,b] \subseteq \bigcup_{i\in I}(c_i,d_i).$$

Assume there does not exist a finite subcover. Split $[a,b] = [a,c] \cup [c,d]$, where $c = (a+b)/2$. Both these subintervals are covered by the cover of $[a,b]$. It cannot be the case that both these subintervals are covered by finite subcovers for the unions of such finite subcovers would give a finite subcover of $[a,b]$. So, take a subinterval that does not have a finite subcover. Split in half again and take one of the new subintervals not covered by a finite subcover.

Continue, in this way finding a sequence of closed intervals

$$J_1 \supseteq J_2 \supseteq J_3 \supseteq ... \quad \text{with} \quad \ell(J_i) = \frac{1}{2^i},$$

none of which have a finite subcover. If a_i is the left hand end point of J_i then $\{a_i\}_{i\geq 1}$ is an increasing sequence bounded above by 1 and so converges, to α say. If b_i is the right hand end point of J_i then $\{b_i\}_{i\geq 1}$ is an decreasing sequence bounded below by 0 and so converges, to β say. Also $|a_i - b_i| = 1/2^i$ for all i and so $\alpha = \beta$. Call this common value γ. Then $\bigcap_i J_i = \{\gamma\}$. Here $\gamma \in [a,b]$ so there exists some $i \in I$ such that $\gamma \in (c_i,d_i)$. Since this is an open interval and $\gamma \in J_i$ for all $i \geq 1$ where $\ell(J_i) \to 0$ as $i \to \infty$, we must have the existence of $n \geq 1$ such that $\gamma \in J_n \subseteq (c_i, d_i)$. But this gives a finite cover of J_n, a contradiction. ■

Theorem 1.4 (Lindelöf's Theorem) *If $\mathcal{G} = \{I_\alpha : \alpha \in A\}$ is a collection of intervals $(a, b) \subseteq \mathbb{R}$, possibly an uncountable collection, then there exists a countable subcollection $\{I_i : i \geq 1\} \subseteq \mathcal{G}$ such that*

$$\bigcup_{\alpha \in A} I_\alpha = \bigcup_{i=1}^{\infty} I_i.$$

Proof Let $x \in \bigcup_{\alpha \in A} I_\alpha$, so there exists $\alpha \in A$ for which $x \in I_\alpha$. If $I_\alpha = (a, b)$, say, then $a < x < b$. Recall that $\mathbb{Q}$ is dense in $\mathbb{R}$ so we can find $r, r' \in \mathbb{Q}$ for which $a < r < x < r' < b$. Write $J = (r, r')$. So for each x we can find an interval J with $x \in J \subseteq I_\alpha$ and the end points of J are rational. The number of rational points is countable so the number of different J that can occur as we vary $x \in \bigcup_{\alpha \in A} I_\alpha$ is countable. So list the J that arise as $J_1, J_2, J_3, ...$. Then

$$\bigcup_{\alpha \in A} I_\alpha \subseteq \bigcup_{i=1}^{\infty} J_i \subseteq \bigcup_{\alpha \in A} I_\alpha,$$

since each $J_i \subseteq I_\alpha$ for some α. Thus

$$\bigcup_{\alpha \in A} I_\alpha = \bigcup_{i=1}^{\infty} J_i.$$

We have seen that, for each $i \geq 1$, we have $J_i \subseteq I_\alpha$ for perhaps many $\alpha \in A$. Just choose one I_α and label it I_i. Then

$$\bigcup_{\alpha \in A} I_\alpha = \bigcup_{i=1}^{\infty} J_i \subseteq \bigcup_{i=1}^{\infty} I_i \subseteq \bigcup_{\alpha \in A} I_\alpha.$$

We must have equality throughout and, hence, the result follows. ■

This proof works because $\mathbb{R}$ contains a countable subset, $\mathbb{Q}$, that is *dense* or, how we have used this above, all open intervals contain an element from this countable subset. A Topological Space, $(X, \mathcal{T})$, that has a countable subset of X with a non-empty intersection with every open set, i.e. set in $\mathcal{T}$, is said to be *seperable.*