Questions

1) Show that if $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X then $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology.

2) Let

$$\mathcal{S} = \{S \subseteq \mathbb{R} : \forall s \in S, \exists \varepsilon > 0 : (s - \varepsilon, s + \varepsilon) \subseteq S\}.$$

Prove

(i) $\mathcal{S} \subseteq \mathcal{U}$ where $\mathcal{U}$ is the usual topology on $\mathbb{R}$.

(ii) $\mathcal{U} \subseteq \mathcal{S}$.

(Hint: show that $\mathcal{S}$ is a topology and then use the minimality of $\mathcal{U}$.)

Deduce that

$$A \in \mathcal{U} \text{ if, and only if, } \forall a \in A, \exists \varepsilon > 0 : (a - \varepsilon, a + \varepsilon) \subseteq A.$$

3) Verify that the co-finite topology **is** a topology on $\mathbb{R}$.

4) Assume that $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{U})$ is continuous by the definition in the notes (so the preimage of an open set is open).

Prove that

$$\forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0 : \forall y \text{ if } |y - x| < \delta \text{ then } |f(y) - f(x)| < \varepsilon.$$

(Hint: Let $x \in \mathbb{R}$ and $\varepsilon > 0$ be given. Then $(f(x) - \varepsilon, f(x) - \varepsilon) \in \mathcal{U}$. So by definition $f^{-1}(f(x) - \varepsilon, f(x) + \varepsilon) \in \mathcal{U}$. Then apply Question 2 to an appropriate point in this preimage.)

5) Let $\mathcal{P}^n$ be the set of all finite cubes in $\mathbb{R}^n$ of the form

$$(a_1, b_1] \times (a_2, b_2] \times ... \times (a_n, b_n].$$

Prove that $\mathcal{P}^2$ is a semi-ring.

Do you think that $\mathcal{P}^n$ is a semi-ring for all $n \geq 1$?

6) Show that $\mathcal{P}^2$ is not a ring.

7) Prove that $\mathcal{R}$ is a ring if, and only if,

(i) $A, B \in \mathcal{R} \Rightarrow A \cap B \in \mathcal{R}$

(ii) $A, B \in \mathcal{R} \Rightarrow A \bigtriangleup B \in \mathcal{R}$ (where $A \bigtriangleup B$ is the symmetric difference, $(A \setminus B) \cup (B \setminus A)$).

8) Let $\mathcal{E}^n$ be the collection of all finite unions of disjoint members of $\mathcal{P}^n$ (and called the elementary figures in $\mathbb{R}^n$).

Prove that $\mathcal{E}^2$ is a ring.

9) Define $\mathcal{F}$, a collection of subsets of $\mathbb{R}$ by $A \in \mathcal{F}$ if, and only if, either $|A|$ is countable or $A = \mathbb{R}$.

If $\mathcal{F}$ a field?

Is $\mathcal{F}$ closed under countable unions?

10) Verify Examples 5(a) and (b) of the notes.

11) Show that the intersection of any non-empty collection of a) rings, b) $\sigma-$rings or c) fields is a a) ring, b) $\sigma-$ring or c) field, respectively.

12) (Theorem 1.7) Let $\mathcal{C}$ be a semi-ring in a set X and $\mathcal{R}(\mathcal{C})$ the ring generated by $\mathcal{C}$ (and so, the intersection of all rings containing $\mathcal{C}$). Let

$$\mathcal{A} = \left\{ A \subseteq X : A = \bigcup_{i=1}^{n} E_i \text{ for some disjoint members } E_i \text{ of } \mathcal{C} \right\}.$$

Prove

(i) $\mathcal{A} \subseteq \mathcal{R}(\mathcal{C})$,

(ii) $\mathcal{A}$ is a ring (so check that $\mathcal{A}$ satisfies the definition and, as in the proof of Corollary 1.5, look at $A \setminus B$ first).

Deduce that $\mathcal{R}(\mathcal{C}) \subseteq \mathcal{A}$.

Hence conclude that $\mathcal{R}(\mathcal{C}) = \mathcal{A}$

13) Let $\mathcal{A}$ be a collection of subsets of X and $\mathcal{F}$ a field of sets from X.

Prove that $\mathcal{A} \subseteq \mathcal{F}$ if, and only if, $\sigma(\mathcal{A}) \subseteq \mathcal{F}$.

14) Show that the Borel sets $\mathcal{B}$ in $\mathbb{R}$ are

(i) generated by all intervals of the form $[a, b]$,

(ii) generated by all intervals of the form $[a, b)$.

Further, show that $\mathcal{B}$ contains

(iii) all one point sets $\{x\}$, $x \in \mathbb{R}$,

(iv) $\mathbb{Q}$,

(v) the set of irrational numbers.

Deduce that

(vi) the co-finite topology on $\mathbb{R}$ is contained in $\mathcal{B}$.

15) Let $F : \mathbb{R} \to \mathbb{R}$ be a distribution function.

Show that the number of points of discontinuity is at most countable. (Hint: try to associate a rational number with each discontinuity).

16) Let $p_n, n \in \mathbb{Z}$ be a sequence of non-negative reals. Let

$$F(x) = \sum_{-\infty < n \leq x} p_n.$$

Show that F is a distribution function.

Why is it important that we have $n \leq x$ in the definition and not $n < x$?

17) Let $X = [0, 1)$ and $\mathcal{C}$ the collection of sets:

$$\phi, X, [0, 1/2), [0, 1/4), [0, 3/4) \qquad \text{and} \qquad [1/4, 3/4).$$

Define μ on $\mathcal{C}$ by:

$$\begin{array}{lll} \mu(\phi) = 0, & \mu(X) = 4, & \mu([0, 1/2)) = 2, \\ \mu([0, 1/4)) = 2, & \mu([0, 3/4)) = 4, & \mu([1/4, 3/4)) = 2. \end{array}$$

Show that μ is additive on $\mathcal{C}$.

Can μ be extended to an additive set function on the ring generated by $\mathcal{C}$?

18) Let $X = \{1, 2, 3, 4, 5\}$ and $\mathcal{C}$ the collection of subsets:

$$\{\phi, X, \{1\}, \{2, 3\}, \{1, 2, 3\}, \{4, 5\}\}.$$

Show that $\mathcal{C}$ is a semi-ring.

Define μ on $\mathcal{C}$ by:

$$\begin{array}{lll} \mu(\phi) = 0, & \mu(X) = 3, & \mu(\{1\}) = 1, \\ \mu(\{2, 3\}) = 1, & \mu(\{1, 2, 3\}) = 2, & \mu(\{4, 5\}) = 1. \end{array}$$

Show that μ is additive.

What is the ring $\mathcal{R}$ generated by $\mathcal{C}$?

Find the additive extension of μ to $\mathcal{R}$ and show that it is a measure.

19) Let X be a uncountable set.

Prove that each of the following functions defined on all subsets of X is an outer measure, and determine the corresponding collection of measurable sets.

(a) $\lambda(\phi) = 0$ and $\lambda(A) = 1$ for all $A \neq \phi$.

(b) $\lambda(\phi) = 0, \lambda(X) = 2$ and $\lambda(A) = 1$ for all $A \neq \phi$ or X.

(c)

$$\lambda(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{if } A \text{ is non-countable.} \end{cases}$$

20) Let $X = \mathbb{N}$ and let $\lambda(E) = n/(n+1)$ if E contains n points and $\lambda(E) = 1$ if E is infinite. Determine the collection of $\lambda-$measurable sets.

21) Let $\mathcal{L}$ be the set of Lebesgue measurable subsets of $\mathbb{R}$.

Prove that, given $E \in \mathcal{L}$ and $\varepsilon > 0$, there exists an open set G, so $G \in \mathcal{U}$, with $E \subseteq G$ and $\mu(G \setminus E) < \varepsilon$.
(Hint. On $\mathcal{L}$ we have that $\mu = \mu^*$, the outer measure that is defined on all subsets of $\mathbb{R}$. Look at the definition of μ^*.)

22) Show that the Lebesgue measure μ on the real line satisfies

$$\mu(cA) = |c|\mu(A)$$

for all $c \in \mathbb{R}$ and all $A \in \mathcal{L}$, the Lebesgue measurable sets.

(Hint. Follow the method of Lemma 2.13 in the notes.)

23) Verify that, given a map $f : X \to Y$ and subsets $A_i \subseteq Y, i \geq 1$, we have

$$f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} f^{-1}\left(A_i\right), \qquad f^{-1}\left(\bigcap_{i=1}^{\infty} A_i\right) = \bigcap_{i=1}^{\infty} f^{-1}\left(A_i\right),$$

and $f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$.

24) (a) Prove that $\{[-\infty, c) : c \in \mathbb{R}\}$ generates $\mathcal{B}^*$.

(b) Prove that $f : X \to \mathbb{R}^*$ is $\mathcal{F}-$measurable if, and only if,

$$\{x : f(x) < c\} \in \mathcal{F}$$

for all $c \in \mathbb{R}$.

25) Let $\{x_n\}$ be a sequence of real numbers. Recall that

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left\{\sup_{r\geq n} x_r\right\} \qquad \text{and} \qquad \liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left\{\inf_{r\geq n} x_r\right\}.$$

Prove that

$$\limsup_{n\to\infty}(-x_n) = -\liminf_{n\to\infty} x_n,$$

26) For a sequence of sets $\{A_n\}$ define

$$\limsup A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k \qquad \text{and} \qquad \liminf A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k.$$

(a) Note that $x \in A_n$ for only finitely many n if, and only if,

$$\exists N \geq 1 : \forall n \geq N x \notin A_n.$$

Negate this statement to get

$$\limsup A_n = \{x : x \in A_n \text{ for infinitely many } n\}.$$

(b) Prove

$$\liminf A_n = \{x : x \in A_n \text{ for all but finitely many } n\}.$$

(c) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4 \subseteq ...$ prove that

$$\limsup A_n = \liminf A_n = \bigcup_{k=1}^{\infty} A_k.$$

(d) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq ...$ prove that

$$\limsup A_n = \liminf A_n = \bigcap_{k=1}^{\infty} A_k.$$

(e) Prove

$$(\limsup A_n)^c = \liminf A_n^c.$$

27) Let $A_n = (-1/n, 1] \subseteq \mathbb{R}$ if n is odd and $A_n = (-1, 1/n] \subseteq \mathbb{R}$ if n is even. Find $\limsup A_n$ and $\liminf A_n$.

28) Let μ be the counting measure on $\mathbb{Z}$ (so $\mu(A) = |A|$ if $A \in \mathbb{Z}$ is finite and $\mu(A) = \infty$ otherwise).

Find a sequence of subsets of $\mathbb{Z}$ satisfying $E_1 \supseteq E_2 \supseteq E_3 \supseteq E_4 \supseteq ...$ (which we say is a *decreasing* sequence) for which $\bigcap_{k=1}^{\infty} E_k = \phi$ but

$$\lim_{n\to\infty} \mu(E_n) \neq 0.$$

29) Let $\mathcal{E}$ be a σ-field and μ a σ-additive set function defined on $\mathcal{E}$ that does not assume the value $-\infty$.

If $E_1 \supseteq E_2 \supseteq E_3 \supseteq E_4 \supseteq ...$ is a nested sequence of members of $\mathcal{E}$ (we say that it is a *decreasing* sequence) with $\mu(E_1) < \infty$, show that

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcap_{k=1}^{\infty} E_k\right).$$

(Hint: Lemma 4.1)

30) Recall that given a Distribution function F, the Lebesgue-Stieltjes measure of $(a, b]$ is given by

$$\mu_F((a,b]) = F(b) - F(a).$$

(a) Give expressions for

$$\mu_F([a,b]), \qquad \mu_F([a,b)) \qquad \text{and} \qquad \mu_F((a,b)).$$

(b) Give an example of a distribution function F such that

$$\mu_F((a,b)) < F(b) - F(a) < \mu_F([a,b])$$

for some $a, b \in \mathbb{R}$.

31) Let

$$F(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1 + x & \text{if } -1 \le x < 0 \\ 2 + x^2 & \text{if } 0 \le x < 2 \\ 9 & \text{if } x \ge 2. \end{cases}$$

Compute the μ_F-measure of each of the following sets:

(a) $\{2\}$, (b) $[-1/2, 3)$, (c) $(-1, 0] \cup (1, 2)$,
(d) $[0, 1/2) \cup (1, 2]$, (e) $\{x : |x| + 2x^2 > 1\}$.

32) Verify parts (i)-(iv) of Theorem 4.2.

33) Apply the Monotone Convergence Theorem to show
(a)

$$\lim_{n\to\infty} n \log\left(1 + \frac{t}{n}\right) = t$$

for all $t \ge 0$. Hence calculate
(b) Deduce

$$\lim_{n\to\infty} \int_0^n \left(1 + \frac{t}{n}\right)^n e^{-2x} dx = 1.$$

34) Apply Corollary 4.13 to show
(a)

$$\int_0^1 \left(\frac{\log x}{1-x}\right)^2 dx = \frac{\pi^2}{3}.$$

(Hint. Expand $(1-x)^{-2}$.)

(b) If $p > -1$,

$$\int_0^1 \frac{x^p \log x}{1-x} dx = -\sum \frac{1}{(p+n)^2}.$$

35) Define $f : [0,1] \to \mathbb{R}$ by $f(x) = 0$ if x is rational and if x is irrational set $f(x) = n$ where n is the number of zeros immediately following the decimal point in the representation of x in the decimal scale.

This function is not simple (it takes infinitely many values) but show how it can be approximated by an increasing sequence of simple measurable functions. Deduce that f is measurable.

Calculate $\int_0^1 f(x)dx$.

36) (Part of Theorem 4.14) Let f be a non-negative $\mathcal{F}$-measurable function on a measure space (X, F, μ).

For $A \subseteq \mathcal{F}$ define

$$\phi(A) = \int_A f d\mu.$$

Let $\{E_n\}$ be collection of disjoint sets from $\mathcal{F}$. Define

$$f_n(x) = \begin{cases} f(x) & \text{if } x \in E_n \\ 0 & \text{if } x \notin E_n, \end{cases}$$

so $f(x) = \sum f_n(x)$ for all $x \in \bigcup_{n=1}^{\infty} E_n$. Using this series representation for f, along with Corollary 4.14, show that

$$\phi\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \phi(E_n),$$

i.e. ϕ is σ-additive.

37) In Fatou's Lemma choose $g_n(x) = 1$ for $x \le n < x+1$ and 0 elsewhere. Show that we then get strict inequality in the result.

(You should remember this example as an aid to remembering which way the inequality goes in Fatou's Lemma.)

38) For $a, b \in \mathbb{R}$ prove

(i)

$$\max(a+b, 0) \le \max(a,0) + \max(b,0),$$

(ii)

$$\min(a+b,0) \geq \min(a,0) + \min(b,0).$$

So, if $f, g : X \to \mathbb{R}^*$ deduce
(iii)

$$(f+g)^+ \leq f^+ + g^+,$$

(iv)

$$(f+g)^- \leq f^- + g^-.$$

39) Show that for reals $a, b \geq 0$ we get equality in $|a-b| \leq a+b$ if, and only if, either $a = 0$ or $b = 0$.

40) (i) Show that if f if $\mathcal{L}$-integrable on $[0, \infty)$ then

$$\int_0^\infty f d\mu = \lim_{n\to\infty} \int_0^n f d\mu$$

(Hint. Use Theorem 4.19.)

(ii) Show that if f if $\mathcal{L}$-measurable on $[0, \infty)$ and non-negative then the same result holds.
(Hint. Use Theorem 4.11.)

41) Prove that $e^{-t}t^{x-1}$ is $\mathcal{L}$-integrable over $(0, \infty)$ for all $x > 0$.

(Hint. First use question 40 to replace the interval of convergence by a finite one. Then for each fixed x find $k : e^{-t}t^{x-1} \leq ke^{-t/2}$ for $t \geq 1$, and so show that the sequence of integrals $\int_0^n f d\mu$ that arise from the use of question 40 is increasing and bounded above.)

42) Define the Gamma function by

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1} d\mu$$

for $x > 0$, which exists by Question 41. Show that

$$\begin{aligned}\Gamma(x) &= \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1} d\mu \\ &= \lim_{n\to\infty} \frac{n! n^x}{x(x+1)...(x+n)},\end{aligned}$$

as long as $-x \notin \mathbb{N} \cup \{0\}$. (Gauss's formula.)

(Hint. Use the ideas of Question 33.)

43) Show that if $a > 1$ then

$$\int_0^\infty \frac{x^{a-1}}{e^x - 1} d\mu = \Gamma(a) \sum_{n=1}^\infty \frac{1}{n^a}.$$

44) Find

(i)

$$\lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \sin(x/n) d\mu,$$

(ii)

$$\lim_{n\to\infty} \int_0^\infty \frac{1 + nx}{(1+x)^n} d\mu.$$

45) Show

(i)

$$\int_0^\infty \text{sech} x^2 d\mu = \sqrt{\pi} \sum_{n=0}^\infty \frac{(-1)^n}{\sqrt{2n+1}}.$$

(You may assume that $\int_0^\infty e^{-x^2} d\mu = \sqrt{\pi}/2$.)

(ii)

$$\int_0^\infty \frac{\cos x}{e^x + 1} d\mu = \sum_{n=1}^\infty \frac{(-1)^{n-1} n}{n^2 + 1}.$$

46) Given $0 < b < a$ evaluate

$$\int_0^\infty \frac{\text{sinh} bx}{\text{sinh} ax} d\mu.$$