Question Sheet 7

1) Let $U = \mathbb{R}$ $P(x, y) : xy = 4$, and $Q(x, y) : x > y$.

Write out the following propositions as mathematical statements. Indicate, with reasons, which are true and which are false. If false, give a counterexample.

(i) $P(8, 0.5)$,

(ii) $\exists y : P(2, y)$,

(iii) $\forall x\ \exists y : P(x, y)$,

(iv) $\exists x\ \forall y : P(x, y)$,

(v) $\forall x\ \forall y : [P(x, y) \rightarrow Q(x, y)]$,

(vi) $\exists x\ \exists y : [P(x, y) \wedge Q(x, y)]$.

2) Negate and simplify the following

(i) $\forall x : x^2 > 0$,

(ii) $\exists x : 2x = 1$,

(iii) $\forall x\ \exists y : x + y = 1$,

(vi) $\forall x\ \forall y : (x > y) \rightarrow (x^2 > y^2)$.

3) Symbolise the following using

$Bx \equiv x$ is beautiful,

$Vx \equiv x$ is valuable,

$Ix \equiv x$ is insured,

(Consider ugly to be the negation of beautiful and worthless the negation of valuable.)

(i) Anything beautiful is valuable,

(ii) If something is insured then it is valuable,

(iii) Not all valuable things are beautiful,

(iv) Some beautiful things are not valuable,

(v) Nothing that is beautiful is insured,

(vi) Only ugly things things are worthless.

4) Negate the following and interpret your answers in English using the definition of the predicates in question 3

(i) $\forall x : Ix \to Vx$,

(ii) $\exists x : Bx \wedge (\neg Vx)$,

(iii) $\forall x : (\neg Vx) \to (\neg Bx)$.

5) Let $U = \mathbb{Z}, P(x) : x > 1$ and $Q(x) : x < 6$.

Determine which of the following are true and which are false.

(i) $\forall x : P(x) \vee Q(x)$,

(ii) $(\forall x : P(x)) \vee (\forall x : Q(x))$,

(iii) $\exists x : P(x) \wedge Q(x)$,

(iv) $(\exists x : P(x)) \wedge (\exists x : Q(x))$.

6) Prove

(i) $\exists x : px \wedge qx \quad \vdash \quad (\exists x : px) \wedge (\exists x : qx)$,

(ii) $(\forall x : px) \wedge (\forall x : qx) \quad \vdash \quad \forall x : px \wedge qx$,

(iii) $\forall x : px \wedge qx \quad \vdash \quad (\forall x : px) \wedge (\forall x : qx)$,

(iv) $\forall x : px \to qx \quad \vdash \quad (\forall x : px) \to (\forall x : qx)$.

7) Prove that the following argument is valid.

$\forall x : p(x) \vee q(x)$,

$\exists x : \neg p(x)$,

$\forall x : (\neg q(x)) \vee r(x)$,

$\forall x : s(x) \to \neg r(x)$

$\vdash \; \exists x : \neg s(x)$.

Hint: use the second premise first and use D.S. twice.

8) Let Ax, Bx and Cx be predicates of one variable, and let u denote an arbitrary element of the universal set U.

(i) Use C.P. to prove

$$\forall x : Ax \to \neg Bx, \quad \forall x : Cx \to Bx \quad \vdash \quad Au \to \neg Cu.$$

(ii) Deduce that

$$\forall x : Ax \to \neg Bx, \quad \forall x : Cx \to Bx \quad \vdash \quad \forall x : Ax \to \neg Cx.$$

9) Symbolise the following argument using

$U =$ set of all animals.

$Wx \equiv x$ is warm blooded,

$Cx \equiv x$ is cold blooded,

$Tx \equiv x$ has no trouble living in a cold climate,

All animals are either warm or cold blooded.

Warm blooded animals have no trouble living in cold climates.

Therefore, the animals that do have trouble living in cold climates are cold blooded.

Prove that the argument is valid.

(Hint use C.P. and D.S.)

10) Is the following valid or invalid?

All law-abiding citizens pay their taxes.

Mr. Blair pays his taxes.

Therefore, Mr. Blair is a law-abiding citizen.

11) Prove, by finding counterexamples involving sets, that the following arguments are **invalid**.

(i) $(\exists x : px) \wedge (\exists x : qx) \quad \vdash \quad \exists x : px \wedge qx,$

Compare with Question 6(i)

(ii) $\exists x : px \wedge bx \quad \vdash \quad \exists x : px \wedge \neg bx,$

(iii) $\forall x : px \to bx \quad \vdash \quad \forall x : bx \to px,$

(iv) $(\forall x : px) \to (\forall x : qx) \quad \vdash \quad \forall x : px \to qx$

Compare with Question 6(iv)