5 Further Set Operations

Definition

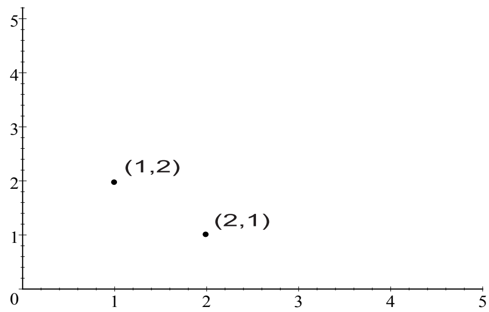
Given two sets A, B and elements $a \in A$, $b \in B$ we can write the *ordered pair* (a, b) where a is the *first coordinate* and b is the *second coordinate.*

Example 54

Let $A = \{d, e, f\}$, $B = \{e, f, g\}$. Then (d, g) is an ordered pair. Note that (f, e) and (e, f) are ordered pairs and if $f \neq e$ they are considered *different* objects. Compare this to sets, where $\{f, e\}$ and $\{e, f\}$ are considered to be the same.

Example 55

Let $A = B = \mathbb{R}$. Then $(1, 2)$ and $(2, 1)$ are ordered pairs. We use them to denote different points in the plane.



Definition

The *Cartesian product* of A and B, written $A \times B$, is the set of all ordered pairs $\{(a, b) \mid (a \in A) \wedge (b \in B)\}$. The set $A \times A$ is often written as A^2.

Example 56

Let $A = \{d, e, f\}$, and $B = \{e, f, g\}$. Then

$$A \times B = \{(d, e), (d, f), (d, g), (e, e), (e, f), (e, g), (f, e), (f, f), (f, g)\},$$

and

$$B \times A = \{(e, d), (e, e), (e, f), (f, d), (f, e), (f, f), (g, d), (g, e), (g, f)\}.$$

So we see that $A \times B \neq B \times A$ in this case, i.e. the Cartesian product is **not** commutative.

Example 57

$\mathbb{R}^2$ = set of all ordered pairs (x, y), $x, y \in \mathbb{R}$; this is the plane.

$\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} = \{((x,y),z) \mid x,y,z \in \mathbb{R}\} = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$; usual 3D-space.

Definition If A is a set then $\mathcal{P}(A)$, the *power set* of A, is the set of all subsets of A.

Example 58

If $A = \{1,2,3\}$ then

$$\mathcal{P}(A) = \{A, \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}.$$

Note we define $\emptyset$ to be a subset of **any** set. So for any set A, $\mathcal{P}(A)$ will always contains $\emptyset$.

Example 59

If $A = \{1\}$, then $\mathcal{P}(A) = \{\emptyset, \{1\}\} = \{\emptyset, A\}$ and $\mathcal{P}(\mathcal{P}(A)) = \{\mathcal{P}(A), \emptyset, \{\emptyset\}, \{A\}\}$. Note $\emptyset \neq \{\emptyset\}$ since we have seen in an earlier example that the set containing an object is different from that object.

Definition If A contains a finite number of distinct elements, then the *cardinality* of A, $\mathrm{Card}(A)$ or $|A|$ denotes the number of distinct elements in A.

Example 60

If $A = \{1,2,3\}$, then $|A| = 3$,

if $B = \{1, \{2,3\}\}$, then $|B| = 2$,

if $C = \{x \in \mathbb{Z} \mid x^2 \leq 4\}$, then $|C| = 5$ since $C = \{-2,-1,0,1,2\}$,

if $D = \{1,2,2,3\}$, then $|D| = 3$,

and if $E = \{1, \{2,3\}, \{2,2,3\}\}$, then $|E| = 2$.

Example 61

If $A = \{1,2\}$, so $|A| = 2$, and $B = \{3,4,5\}$, so $|B| = 3$, then $A \cup B = \{1,2,3,4,5\}$, so $|A \cup B| = 5 = |A| + |B|$.

But if $C = \{a,b,c\}$, when $|C| = 3$, and $D = \{c,d,e\}$, when $|D| = 3$, then $C \cup D = \{a,b,c,c,d,e\}$, and $|C \cup D| = 5 \neq |C| + |D|$.

In $|C|+|D|$ we are counting the element c twice; i.e. counting the elements of $C \cap D$ twice. In fact

$$|C| + |D| - |C \cap D| = 3 + 3 - 1 = 5 = |C \cup D|.$$

This works in general.

Addition Rule

For finite sets, $|A \cup B| = |A| + |B| - |A \cap B|$.

Definition If $A \cap B = \phi$ we say A and B are *disjoint.*

So, if A and B are disjoint then $|A \cup B| = |A| + |B|$.

***Example 62**

Out of 50 students, 30 are studying BASIC, 25 are studying PASCAL and 10 both. How many students study either language?

We have $U =$ the set of these 50 students, $|B| = 30$, $|P| = 25$ and $|B \cap P| = 10$. We want to know $|B \cup P|$ and so we apply the addition rule.

Therefore the answer is $|B \cup P| = |B| + |P| - |B \cap P| = 30 + 25 - 10 = 45$.*

***Example 63**

If there are 25 elements in either A or B with 10 in B and 5 in both, how many are in A?

For this we do not apply the addition rule directly but first we rearrange it as:

$|A| = |A \cup B| - |B| + |A \cap B| = 25 - 10 + 5 = 20$.*

***Example**

(i) How many integers between 1 and 1000 (inclusive) start with 7?

Answer: Simply count them. 7, 70, 71,..., 79, 700, 701,..., 799. Thus a total of 1+10+100=111.

(ii) How many integers between 1 and 1000 (inclusive) end with 7?

Answer: Simply count them. 7, 17, 27,..., 97, 107, 117,..., 987, 997. Think about these as

$$007, 017, 027, ..., 107, 117, ..., 987, 997,$$

which are the same in number as in the list

$$00, 01, 02, ..., 10, 11, ..., 98, 99,$$

i.e. a 100.

(iii) How many integers between 1 and 1000 (inclusive) either begin or end with 7?

Answer: Let U be the set of integers between 1 and 1000.

Let $A \subseteq U$ be the set of integers starting in 7.

Let $B \subseteq U$ be the set of integers ending in 7.

We want $|A \cup B|$. To use the addition rule we need $|A \cap B|$, the number of integers both starting and ending in 7. We count these, 7, 77, 707, 717, ..., 797. So there are 12 of these.

Thus $|A \cup B| = |A| + |B| - |A \cap B| = 111 + 100 - 12 = 199$.

***Example** How may integers between 1 and 99 inclusive are divisible by either 2 or 3 or both?

To solve this let the universe be given as $U = \{n \in \mathbb{N} : 1 \leq n \leq 99\}$.

Then define

$$S_2 = \{n : \ 2 \text{ divides } n\} \quad \text{and} \quad S_3 = \{n : \ 3 \text{ divides } n\}.$$

We need to calculate $|S_2 \cup S_3|$.

From the addition rule we have $|S_2 \cup S_3| = |S_2| + |S_3| - |S_2 \cap S_3|$. It is easily seen that

$$|S_2| = \frac{98}{2} = 49 \quad \text{and} \quad |S_3| = \frac{99}{3} = 33.$$

For $|S_2 \cap S_3|$ we note that

$$\begin{aligned} S_2 \cap S_3 &= \{n : \ (2 \text{ divides } n) \wedge (3 \text{ divides } n)\} \\ &= \{n : \ 6 \text{ divides } n\} \\ &= S_6, \text{ say.} \end{aligned}$$

Then

$$|S_2 \cap S_3| = |S_6| = \frac{96}{6} = 16.$$

Hence our result is

$$|S_2 \cup S_3| = |S_2| + |S_3| - |S_2 \cap S_3| = 49 + 33 - 16 = 66.$$

What would be the answer to the question of how may integers between 1 and 99 inclusive are divisible by either 2, 3 or 5? (That is, calculate $|S_2 \cup S_3 \cup S_5|$). The following result may be of use.*

Example 64 If we have three sets A, B and $C \subseteq U$, then

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad -|A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|. \end{aligned}$$

*To see this, start with

$$\begin{aligned} |A \cup B \cup C| &= |(A \cup B) \cup C| \\ &= |A \cup B| + |C| - |(A \cup B) \cap C| \\ &= (|A| + |B| - |A \cap B|) + |C| - |(A \cup B) \cap C|. \end{aligned} \tag{1}$$

having used the addition rule twice. But, by the distributive law, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$, which again is a union to which we can apply the addition rule. This gives

$$\begin{aligned} |(A \cup B) \cap C| &= |(A \cap C) \cup (B \cap C)| \\ &= |(A \cap C)| + |(B \cap C)| - |(A \cap C) \cap (B \cap C)| \\ &= |A \cap C| + |B \cap C| - |A \cap B \cap C|. \end{aligned} \tag{2}$$

Combining (1) and (2) gives

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad -|A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|. \end{aligned}$$

So as remarked this could be used to calculate $|S_2 \cup S_3 \cup S_5|$. (I leave it to the student to show the answer is 74.)*

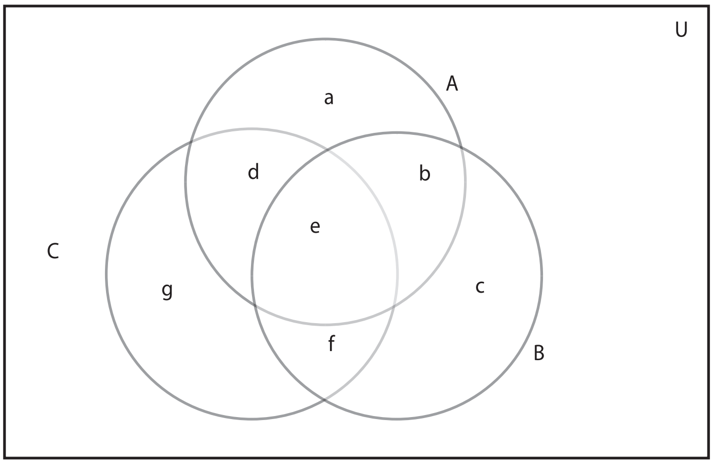
Of course it is not necessary to use a formula, and when we have two or three sets it often convenient to use Venn diagrams as in the next example.

Example Assume we have three sets P, Q and $R \subseteq U$ and we know the following:

$$\begin{array}{ll} |P \cup Q \cup R| = 16, & |R \setminus (P \cap Q)| = 8, \\ |Q \setminus R| = 5, & |(P \cup Q) \cap R^c| = 6, \\ |(P \cup R) \setminus Q| = 5, & |R \cap A^c| = 5. \\ |(P \cap R) \cup Q| = 11, & \end{array}$$

Find $|P|, |Q|$ and $|R|$.

Solution Consider



We have seven regions $a - g$ and we have been given seven "bits" of information.

$$
\begin{aligned}
&|P \cup Q \cup R| = 16 && \Rightarrow a+b+c+d+e+f+g = 16 && (1)\\
&|Q \setminus R| = 5 && \Rightarrow b+c = 5 && (2)\\
&|(P \cup R) \setminus Q| = 5 && \Rightarrow b+e+f = 5 && (3)\\
&|(P \cap R) \cup Q| = 11 && \Rightarrow b+c+d+e+f = 11 && (4)\\
&|R \setminus (P \cap Q)| = 8 && \Rightarrow d+f+g = 8 && (5)\\
&|(P \cup Q) \cap R^c| = 6 && \Rightarrow a+b+c = 6 && (6)\\
&|R \cap A^c| = 5 && \Rightarrow f+g = 5 && (7)
\end{aligned}
$$

We have to solve this "system" of equations.

(5)-(7) $\Rightarrow d = 3$.
(6)-(5) $\Rightarrow a = 1$.
(1)-(6)-(5) $\Rightarrow e = 16 - 6 - 8 = 2$.
(4)-(2) $\Rightarrow d + e + f = 6$, and so $f = 1$.
(7) $\Rightarrow f + g = 5$, and so $g = 4$.
(3) $\Rightarrow b + 2 + 1 = 5$ and so $b = 2$.
(2) $\Rightarrow c = 3$.

Hence

$$
\begin{aligned}
|A| &= a+b+d+e = 8\\
|B| &= b+c+e+f = 8\\
|C| &= d+e+f+g = 10.
\end{aligned}
$$

*__Example__ There are 70 students making third year option choices that may include CT314, CT322 and CT343.

19 students chose neither CT322 nor CT343.

13 students chose CT314 but not CT322.

17 students chose both CT314 and CT343.

19 students chose both CT322 and CT343.

Of the students not choosing CT314, 28 have chosen either CT322 or CT343.

13 students chose CT322 but not CT343.

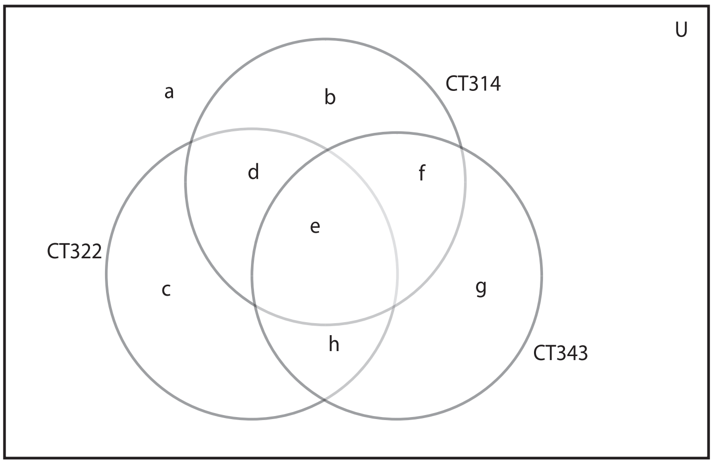
11 students chose CT343 without choosing either of CT314 or CT322.

How many students do each course?

How many students do none of the 3 courses?

Note that the English expression of "neither A nor B" is translated as "not $(A$ or $B)$", i.e. $\neg(A \vee B)$.

To solve the problem, draw the following diagram.



Let the cardinality of the set represented by a region be given by the lower case letter in that region. In terms of the diagram, the information above becomes

$$70 = a + b + c + d + e + f + g + h$$

along with

$$\begin{array}{lll} 19 = a + b, & 13 = b + f, & 17 = e + f, \\ 19 = e + h & 28 = c + h + g, & 13 = c + d, \\ 11 = g. & & \end{array}$$

These can be solved to give, first,

$$70 = (a + b) + d + (e + f) + (c + g + h) = d + 64$$

so $d = 6$.

Then $c = 13-6 = 7$, $h = 28-7-11 = 10$, $e = 19-10 = 9$, $f = 17-9 = 8$, $b = 13 - 8 = 5$ and $a = 19 - 5 = 14$.

Thus the number of students on CT314 is $a+d+e+f = 14+6+9+8 = 37$.

And the number of students on CT322 is $c+d+e+h = 7+6+9+11 = 33$.

While the number of students on CT343 is $e+f+g+h = 9+8+11+10 = 38$.

And 14 do none of the three courses.

*__Example 65__ Let A, B and $C \subseteq U$, a universal set.
Assume that $|U| = 24$ with

$$\begin{array}{lll} |A| = 8, & |B| = 13, & |C| = 13, \\ |A \cap B| = 5, & |A \cap C| = 3, & |B \cap C| = 6, \\ |A \cap B \cap C| = 2. & & \end{array}$$

How many elements are in A that are in no other set? (i.e. $(A \cap B^c \cap C^c)$),
How many elements are in no set? (i.e. $(A \cup B \cup C)^c$).
How many elements are not in C? (i.e. C^c).

This is most easily solved by drawing a Venn diagram of 3 overlapping sets and writing in each region the number of elements in the set represented by that region. (When filling in a Venn diagram always start in the middle. In this case start with $A \cap B \cap C$.)
The answers are 2, 2 and 11 respectively.*

It is not feasible to use Venn diagrams when you have four or more sets and so you have to use a formula. But from the addition rule and Example 64 you should be able to see a pattern and guess that

$$|A \cup B \cup C \cup D| = |A| + |B| + |C\ | + |D|$$

$$\begin{aligned} &-|A \cap B| - |A \cap C\ | - |A \cap D| - |B \cap C\ | - |B \cap D| - |C \cap D| \\ &+|A \cap B \cap C\ | + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ &-|A \cap B \cap C \cap D|. \end{aligned}$$

Multiplication Rule

Assume A and B are finite and non-empty, then $|A \times B| = |A|.|B|$.

*To see this, assume $A = \{a_1, ..., a_m\}$ and $B = \{b_1, ..., b_n\}$. Then $A \times B$ can be represented by a diagram

$$\left\{ \begin{array}{cccccc} (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & \cdots & \cdots & (a_1, b_n) \\ (a_2, b_1) & (a_2, b_2) & & & & (a_2, b_n) \\ (a_3, b_1) & & & & & \vdots \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ (a_m, b_1) & \cdots & \cdots & \cdots & \cdots & (a_m, b_n) \end{array} \right\}.$$

There are no duplicated ordered pairs in this set and $|A \times B|$, the number of ordered pairs, is equal to

$$\begin{aligned} \text{number of rows} \times \text{number of columns} &= m \times n \\ &= |A| \times |B|. \end{aligned}$$

Cardinality of the Power Set

If $|A| = n$ then $|\mathcal{P}(A)| = 2^n$.

So if one element is added to a set the number of subsets doubles.

*For a possible proof consider first $|A| = 3$ and write $A = \{a, b, c\}$, say.

List the subsets of A in a special way, as

$$\begin{array}{llll} \phi = \{\_, \_, \_\}, & \{a\} = \{a, \_, \_\}, & \{b\} = \{\_, b, \_\}, & \{c\} = \{\_, \_, c\}, \\ \{a, b\} = \{a, b, \_\}, & \{a, c\} = \{a, \_, c\}, & \{b, c\} = \{\_, b, c\}, & A = \{a, b, c\}. \end{array}$$

We now give a correspondence

$$\begin{array}{ll} \{\_, \_, \_\} \leftrightarrow 000, & \{a, b, \_\} \leftrightarrow 110, \\ \{a, \_, \_\} \leftrightarrow 100, & \{a, \_, c\} \leftrightarrow 101, \\ \{\_, b, \_\} \leftrightarrow 010, & \{\_, b, c\} \leftrightarrow 011, \\ \{\_, \_, c\} \leftrightarrow 001, & \{a, b, c\} \leftrightarrow 111. \end{array}$$

Hopefully you can see how this correspondence is defined. But it shows there are as many subsets as there are binary numbers between 000 and 111, inclusive, i.e. 2^3 of them.

It should not be hard to show that this holds in general.*