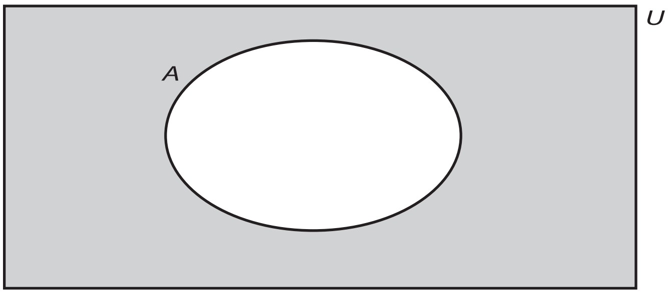
3.4 Set Operations

Given a set A, the *complement* (in the Universal set U) A^c is the set of all elements of U that are not in A. So $A^c = \{x \mid x \notin A\}$.



(This type of picture is called a *Venn diagram.*)

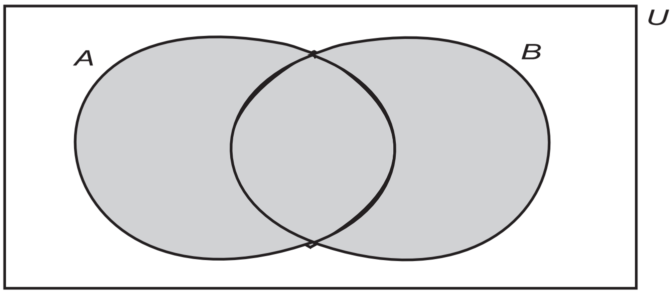
Example 39 Let $A = \{1, 2, 3\}$.

If $U = \mathbb{N}$ then $A^c = \{4, 5, 6, ...\}$.

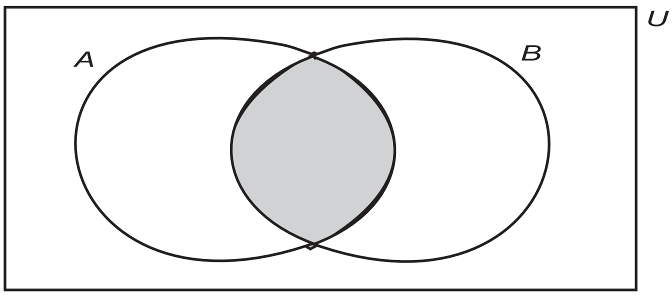
If $U = \mathbb{Z}$ then $A^c = \{..., -2, -1, 0, 4, 5, ...\}$.

So, again, the Universal set is important.

Let A and B be sets. The *union* of A and B is the set of elements belonging to either A or B: $A \cup B = \{x \mid (x \in A) \vee (x \in B)\}$.

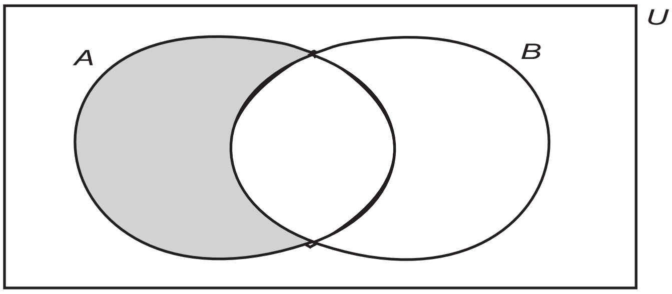


The *intersection* of A and B is the set of elements belonging to both A and B: $A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}$.



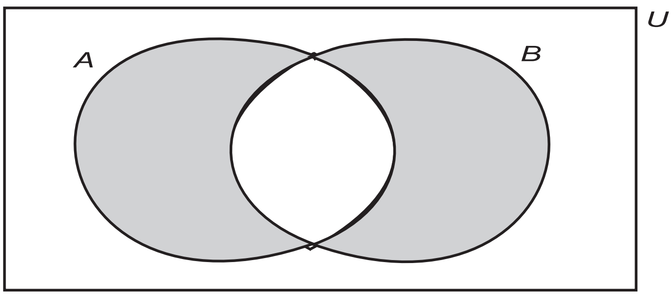
The *difference* of A and B is the set of elements of A that do **not** belong to B:

$$\begin{aligned} A \setminus B &= \{x \mid (x \in A) \wedge (x \notin B)\} \\ &= \{x \mid (x \in A) \wedge (x \in B^c)\} \text{ by definition of complement} \\ &= A \cap B^c \text{ by definition of intersection.} \end{aligned}$$



Note that in general it is **not** true that $A \setminus B$ and $B \setminus A$ are equal.
The *symmetric difference* of A and B is the set of elements that belong to A or B **but not both**:

$$\begin{aligned} A \triangle B &= \{x \mid (x \in A \ \vee \ x \in B) \wedge (x \notin (A \cap B))\} = (A \cup B) \setminus (A \cap B) \\ &= (A \cup B) \cap (A \cap B)^c \text{ by above result on difference.} \end{aligned}$$



* Note that $A \triangle B$ can easily be expressed in terms of the exclusive or, $\veebar$, defined in Question 5 Sheet 2. For

$$A \triangle B = \{x \mid \ (x \in A) \veebar (x \in B))\}$$

3.5 Set Laws

Recall $A = B$ if, and only if, $A \subseteq B$ and $B \subseteq A$.

To prove $A \subseteq B$ use proof by the *pick-a-point* method; i.e. take an arbitrary element $u \in U$ and assume $u \in A$. Then use any known true statements, including properties of A and B to prove $u \in B$.

Example 40

Let $U = \mathbb{Z}$, and let $P(x)$ be the predicate $|x-1| \leq 2$ with $R(x)$ the predicate $|x| \geq 2$.

Prove that $\{x \mid \neg R(x)\} \subseteq \{x \mid P(x)\}$.

Proof. Take any $u \in \{x \mid \neg R(x)\}$. This means that $\neg R(u)$ is TRUE, i.e. $R(u)$ is FALSE. Thus $|u| < 2$. But $u \in U = \mathbb{Z}$. So $|u| < 2$ means that $u = -1, 0$ or 1. For these u we can calculate $|u-1|$, obtaining the values $2, 1$ or 0. In all cases $|u-1| \leq 2$ and so $P(u)$ is true. Hence $u \in \{x \mid P(x)\}$. This is true for all $u \in \{x \mid \neg R(x)\}$ hence $\{x \mid \neg R(x)\} \subseteq \{x \mid P(x)\}$. $\blacksquare$*

Note: If we are given $A, B \subseteq U$ we can try to show $A = B$ by showing that the propositions $u \in A$ and $u \in B$ are equivalent, i.e. $(u \in A) \equiv (u \in B)$ for all $u \in U$.

*To see why this suffices, assume we have managed to show that $(u \in A) \equiv (u \in B)$ for all $u \in U$. Then if $u \in A$ is true we must have that $u \in B$ is true, which is the definition of $A \subseteq B$. Similarly, if $u \in B$ is true then $u \in A$ is also true and so $B \subseteq A$. Hence $A = B$.

As an example we can try to prove $(A \cup B)^c = A^c \cap B^c$.

Example Use the Boolean Laws for *logic* to prove $(A \cup B)^c = A^c \cap B^c$.

(If you are asked to use the laws for logic you *cannot* use the laws for sets!)

Take any $u \in U$. Then

$$
\begin{aligned}
u \in (A \cup B)^c \quad &\equiv u \notin A \cup B \\
&\qquad\qquad \text{(By definition of } ^c\text{)} \\
&\equiv \neg\,(u \in A \cup B) \\
&\qquad\qquad \text{(By definition of } \notin\text{)} \\
&\equiv \neg\,((u \in A) \;\vee\; (u \in B)) \\
&\qquad\qquad \text{(By definition of } \cup\text{)} \\
&\equiv (\neg\,(u \in A) \wedge \neg\,(u \in B)) \\
&\qquad\qquad \text{(De Morgan's law \textit{for logic})} \\
&\equiv u \notin A \;\wedge\; u \notin B \\
&\qquad\qquad \text{(By definition of } \notin\text{)} \\
&\equiv u \in A^c \;\wedge\; u \in B^c \\
&\qquad\qquad \text{(By definition of } ^c\text{)} \\
&\equiv u \in A^c \cap B^c \\
&\qquad\qquad \text{(By definition of } \cap\text{)}
\end{aligned}
$$

Thus we see how one of De Morgan's law *for logic* gives one of the two **De Morgan's Laws for sets:**

(a) $(A \cap B)^c = A^c \cup B^c$,
(b) $(A \cup B)^c = A^c \cap B^c$.

Similarly we can prove the distributive law $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, by making use of the distributive law for *propositions*.

Example Use the Boolean Laws for *logic* to prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Take any $u \in U$. Then

$$\begin{aligned}
u \in A \cup (B \cap C) &\equiv (u \in A) \vee (u \in B \cap C) \\
&\qquad \text{by definition of } \cup \\
&\equiv (u \in A) \vee ((u \in B) \wedge (u \in C)) \\
&\qquad \text{by definition of } \cap \\
&\equiv ((u \in A) \vee (u \in B)) \wedge ((u \in A) \vee (u \in C)) \\
&\qquad \text{distributive law for } \textit{propositions} \\
&\equiv (u \in A \cup B) \wedge (u \in A \cup C) \\
&\qquad \text{by definition of } \cup \\
&\equiv u \in (A \cup B) \cap (A \cup C) \\
&\qquad \text{by definition of } \cap.
\end{aligned}$$

This is one of the results in

The Boolean Laws for Sets:

Assume $A, B, C \subseteq U$, the universal set.

1) a) $A \cup B = B \cup A$
   b) $A \cap B = B \cap A$ } Commutative laws

2) a) $(A \cup B) \cup C = A \cup (B \cup C)$
   b) $(A \cap B) \cap C = A \cap (B \cap C)$ } Associative laws

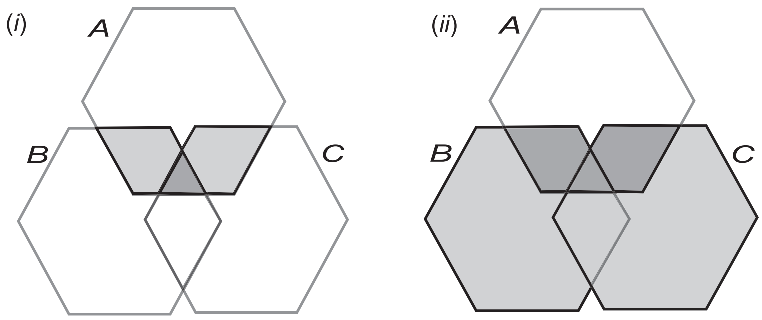
3) a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ } Distributive laws

4) a) $A \cup \emptyset = A$
   b) $A \cap U = A$

5) a) $A \cup A^c = U$
   b) $A \cap A^c = \emptyset$.

We can use Venn diagrams to see that these results are reasonable though you **cannot** use Venn diagrams to prove these results.

Example 41



In (i) the set $(A \cap B) \cup (A \cap C)$ is represented by all the shaded region while in (ii) the set $A \cap (B \cup C)$ is represented by the darker shaded regions. So $(A \cap B) \cup (A \cap C)$ and $A \cap (B \cup C)$ have the same diagrams.

Proofs can be given for all the laws though some are a little tricky.

Example (i) Take any $u \in U$. Then $u \in A \cup \emptyset \equiv (u \in A) \vee (u \in \emptyset)$. But $\emptyset$ is the empty set so $u \in \emptyset$ is false for all possible $u \in U$, i.e. $u \in \emptyset$ is a contradiction or, in symbols, $(u \in \emptyset) \equiv O$. Thus

$$\begin{aligned} u \in A \cup \emptyset &\equiv (u \in A) \vee (u \in \emptyset) && \text{definition of } \cup \\ &\equiv (u \in A) \vee O \\ &\equiv u \in A && \text{law 4a for } \textit{propositions.} \end{aligned}$$

Hence we have used law 4a for *logic* to prove law 4a for *sets.*

(ii) Note, since in any given problem all elements lie in the universal set U, the proposition $u \in U$ is trivially true for all u, i.e. $(u \in U) \equiv I$. For an example take any $u \in U$. Then

$$\begin{aligned} u \in A \cup A^c &\equiv (u \in A) \vee (u \in A^c) && \text{definition of } \cup \\ &\equiv (u \in A) \vee (u \notin A) && \text{definition of complement} \\ &\equiv (u \in A) \vee (\neg\ (u \in A)) && \text{definition of } \notin \\ &\equiv I && \text{law 5a for } \textit{propositions} \\ &\equiv u \in U && I \equiv (u \in U) \text{ seen above.} \end{aligned}$$

Thus $A \cup A^c = U$. Hence we have used law 4a for *logic* to prove law 5a for *sets.*

Hopefully you can now see why the laws for sets are identical to the laws for propositions.

The laws can be used to simplify expressions:

Example 42 (i)

$$
\begin{aligned}
&(C^c \cap A \cap B) \cup (C^c \cap A^c \cap B) \\
&\qquad = C^c \cap ((A \cap B) \cup (A^c \cap B)) && \text{(law 3b), distributive} \\
&\qquad = C^c \cap ((A \cup A^c) \cap B) && \text{(law 3b), distributive} \\
&\qquad = C^c \cap (U \cap B) && \text{(law 5a)} \\
&\qquad = C^c \cap B && \text{(law 4b)}
\end{aligned}
$$

Compare with Ex 11.

*(ii) By definition $A \triangle B = (A \cup B) \setminus (A \cap B)$. From the Venn diagram it looks as if this should equal $(A \setminus B) \cup (B \setminus A)$. Can you use the laws to prove this?

$$
\begin{aligned}
A \triangle B &= (A \cup B) \setminus (A \cap B) && \text{(By definition of } \triangle) \\
&= (A \cup B) \cap (A \cap B)^c && \text{(By definition of } \setminus) \\
&= (A \cup B) \cap (A^c \cup B^c) && \text{(By DeMorgan's law)} \\
&= ((A \cup B) \cap A^c) \cup ((A \cup B) \cap B^c) \\
& && \text{(Distributive law)} \\
&= \{(A \cap A^c) \cup (B \cap A^c)\} \cup \{(A \cap B^c) \cup (B \cap B^c)\} \\
& && \text{(Distributive law, again)} \\
&= \{\phi \cup (B \cap A^c)\} \cup \{(A \cap B^c) \cup \phi\} && \text{(law 5b)} \\
&= \{(B \cap A^c)\} \cup \{(A \cap B^c)\} && \text{(law 4a)} \\
&= \{(B \setminus A)\} \cup \{(A \setminus B)\} && \text{(By definition of } \setminus) \\
&= (A \setminus B) \cup (B \setminus A) && \text{(law 1b)}
\end{aligned}
$$

Note how we have had to use the distributive laws to make the expressions "more complicated" before simplification.

*Additional Material

(Not covered in lectures) Just as for propositions it is possible using these laws (along with De Morgan's laws) to prove the following "obvious" results, seen earlier for propositions at the end of section 1.2.2.

$$(A^c)^c = A,$$

$$\left.\begin{aligned} A \cup U &= U \\ A \cap \phi &= \phi. \end{aligned}\right\} \qquad \text{(Domination Laws)}$$

$$\left.\begin{aligned} A \cup A &= A \\ A \cap A &= A \end{aligned}\right\} \qquad \text{(Idempotent Laws)}$$