3 Sets

3.1 Definition

A *set* is a collection of objects. These objects are called *elements* or *members* of the set.

Example 30

The set of houses in London.
The set of books in the Library.
The Greek alphabet.
The set of natural numbers with digits 0 or 1.

Notation

Usually A, B,..., X, Y will denote sets and $a, b, ..., x, y, ...$ will denote elements. If a is an element of A, we write $a \in A$. If a is *not* an element of A write $a \notin A$.

Note that "$a \in A$" is an assertive sentence; it is asserting that the object a is in the set A.

Assumption 1

We only consider as sets those collections A such that, given an object a, we can say whether the sentence "$a \in A$" is true or false, i.e. $a \in A$ is a proposition.

Definition

Let A and B be sets. Then A is a *subset* of B if every element of A is in B. Write $A \subseteq B$.

If A is a subset of B but there exist elements of B that are *not* elements of A we say that A is a *proper subset* of B and write $A \subset B$.

Assumption 2

In any given problem we assume that the sets under consideration are all subsets of some Universal set U.

Definition

We say that two sets are *equal* and write $A = B$ if, and only if $A \subseteq B$ and $B \subseteq A$.

Example 31 Let $U = \mathbb{N}$.

(i) Obviously, $\{1, 4, 4, 3\} \subseteq \{1, 4, 3\}$ and $\{1, 4, 3\} \subseteq \{1, 4, 4, 3\}$.
So $\{1, 4, 4, 3\} = \{1, 4, 3\}$.

This shows that we can disregard duplicated elements.

(ii) Checking the definition, we can see that $\{1, 2, 3\} = \{3, 2, 1\}$.

This shows that the order of the elements in a set is unimportant.

Example 32 Let $U = \mathbb{N}$. Is $\{1\} = 1$? NO.

* We try to see if the definition of equality is satisfied, i.e. we check to see if both $\{1\} \subseteq 1$ and $1 \subseteq \{1\}$ hold.

We check $\{1\} \subseteq 1$ by taking an element of $\{1\}$, and we have no choice we must take 1, and seeing if this is an element of the right hand side, i.e. 1. So $1 \in 1$? This is obviously false, so $\{1\} \nsubseteq 1$ and thus $\{1\} \neq 1$.*

In general an object is different to the set containing it.

3.2 Denoting a Set

(a) **List** e.g. $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ or {a, e, i, o, u},

(b) **Pattern** e.g. $\{1, 2, 3, ...\}$ (natural numbers, denoted $\mathbb{N}$),

$\{\ ...-2, -1, 0, 1, 2, ...\}$ (integers, denoted $\mathbb{Z}$)

Note that $\mathbb{N} \subseteq \mathbb{Z}$ and both of these sets are subsets of $\mathbb{R}$, the set of all *real numbers*.

(c) **Predicate Form** for this we need a definition.

Definition

A *predicate* is an assertion that contains one or more variables such that, if the variables are replaced by objects from a given Universal set U, then we obtain a proposition.

*Sometimes a predicate is called an *open statement*, the variables are called *free variables* and U is called the *Universe of discourse*.

Example 33

(i) The predicate $x + y \geq 10$, $(U = \mathbb{Z})$ has two variables x, y. So, for example, if we choose $x = -1$, $y = 12$ $(\in U)$ then we get the proposition $-1 + 12 \geq 10$ which is TRUE.

(ii) The predicate $z \in \{1, 2, 3, 4\}$, $(U = \mathbb{N})$ has one variable, z. So, for example, if we choose $z = 5$ $(\in U)$ we get the proposition $5 \in \{1, 2, 3, 4\}$ which is FALSE.

Definition

If $p(x)$ is a predicate with one variable, and U is a universal set then those $u \in U$ for which $p(u)$ is true is the *solution set*. Write it as $\{x \mid p(x)\}$ or $\{x \ : \ p(x)\}$.

Example 34

(i) Let U = Set of all letters of the alphabet. Then

$$\{x \mid x \text{ is a vowel}\} = \{\text{a, e, i, o, u}\}.$$

(ii) Let U = set of real numbers (denoted by $\mathbb{R}$). Then

$$\begin{aligned}
&\left\{x : x = \frac{p}{q} \text{ for some } p, \in \mathbb{Z}, q \in \mathbb{N}\right\} \\
= &\left\{0, -\frac{1}{1}, \frac{1}{1}, -\frac{2}{1}, \frac{2}{1}, -\frac{3}{1}, \frac{3}{1}, \cdots \right. \\
&\quad -\frac{1}{2}, \frac{1}{2}, -\frac{2}{2}, \frac{2}{2}, -\frac{3}{2}, \frac{3}{2}, \cdots \\
&\quad\; -\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, \frac{2}{3}, -\frac{3}{3}, \frac{3}{3}, \cdots \\
&\quad\;\; \left. -\frac{1}{4}, \frac{1}{4}, -\frac{2}{4}, \frac{2}{4}, -\frac{3}{4}, \frac{3}{4}, \cdots, \cdots, \cdots \right\}
\end{aligned}$$

is the set of rational numbers (or fractions or quotients) denoted by $\mathbb{Q}$. Note that $\mathbb{Z} \subseteq \mathbb{Q}$.

(iii) Let $U = \mathbb{N}$. Then

$$\{x \mid x^2 = 3\} \text{ is empty.}$$

That is, there are no natural numbers x satisfying $x^2 = 3$. The empty set is always denoted by $\emptyset$.

(iv) Let $U = \mathbb{N}$. Then

$$\{x \mid x^2 = 4\} = \{2\}.$$

(v) Let $U = \mathbb{Z}$. Then

$$\{x \mid x^2 = 4\} = \{-2, 2\}.$$

From the last two examples we see that the Universal set is important.

(d) **Inductive form**

Definition

A set is defined *inductively* if we have three steps:

(1) A list or description of some elements in the set (so the set is non-empty).

(2) A procedure for obtaining new elements from elements known to be in the set. This is applied iteratively (i.e. again and again).

(3) A claim that only elements obtained from (1) and (2) occur in the set.

Example 35

Define $A \subseteq \mathbb{Z}$ inductively by:

(1) 4, 6 are in A

(2) If x and y are any elements of A, then $x+y$ and $x-y$ are also elements of A.

(3) Only numbers obtained by (1) and (2) are in A.

Applying (2) just once we have 4 choices for the pair (x, y), i.e. $(4, 4)$; $(4, 6)$; $(6, 4)$ and $(6, 6)$. Thus we get the new elements as shown in the table

x	y	$x+y$	$x-y$
4	4	8	0
4	6	10	-2
6	4	10	2
6	6	12	0

So, after one application of (2), we find that $\{-2, 0, 2, 4, 6, 8, 10, 12\} \subseteq A$.

After two applications $\{-14, -12, ..., 22, 24\} \subseteq A$.

If we continue in this way, we get $\{..., -6, -4, -2, 0, 2, 4, ...\} \subseteq A$, i.e. A contains all even integers. But by step (3) we know there are no other elements in A. Hence A is the set of all even integers (often written as $2\mathbb{Z}$).

(We have only shown that A contains even integers. Can you show that A contains any even integer you might be given? i.e. A **is** the set of all even integers?)

3.3 Formal Languages

Let E be a finite set of symbols, the *alphabet*. A *word* over E is a finite sequence of symbols of E. The word with no symbols, the empty word, is denoted by Λ. The collection of all words over E is denoted E^*. Any subset $L \subseteq E^*$ is a *formal language*.

Note, in English we are interested in both the meaning of words (the *semantics* of the language) and how the words are put together (the *syntax* of the language). For formal languages there is no concept of the meaning of the words.

Example 36

Let $E = \{0, 1\}$. Then $E^* = \{\Lambda, 0, 1, 00, 01, 10, 11, 000, ...\}$. So a possible language is $\{1, 11, 111, ...\}$.

Given $w \in E^*$, it can be difficult to check whether $w \in L$. It is easier if L is inductively defined.

Example 37

Let $E = \{a, b\}$. Define L by

(1) $a \in L$

(2) If $x \in L$ with $x = ya$ for some $y \in E^*$ (i.e. the word x ends in the symbol a), then $xb \in L$.

If $x \in L$ with $x = yab$ for some $y \in E^*$(i.e. the word x ends in ab), then $xba \in L$.

(3) Only words obtained in (1) and (2) are in L.

Then $L = \{a, ab, abba, abbab, abbabba, ...\}$.

Is $abbaabbabba \in L$? Give your reasons.

* Propositional Logic as a formal Language (None of the following will be examined.)

Example 38

Let $E = \{p,\ q,\ (,\ ),\ \neg,\ \wedge,\ \vee,\ \ \rightarrow,\ \ \leftrightarrow\}$. Define L by:

(1) $p, q \in L$

(2) If $\alpha, \beta \in L$, then so are

$$\neg(\alpha)$$
$$(\alpha) \vee (\beta)$$
$$(\alpha) \wedge (\beta)$$
$$(\alpha) \rightarrow (\beta)$$
$$(\alpha) \leftrightarrow (\beta).$$

(3) Only expressions obtained in (1) and (2) are in L.

Here E^* will contains many words, such as $\rightarrow)$, $((p \rightarrow$, $(p) \wedge (q)$, $pq \rightarrow$, $q\neg p\neg q$, ... But L will contain words such as $(\neg(p)) \vee (q)$, $\neg((p) \rightarrow (\neg q))$, $((p) \leftrightarrow (q)) \rightarrow (\neg(q))$, ... These words of L look very reminiscent of propositional forms. But as elements of a *formal* language the words have no such meaning. **If** we did look upon p and q as logical variables **then** we would be looking on L as consisting of propositional forms in p and q. But further, **if** we gave truth values to p and q **then** we could find the truth values of each of the words in L. (We just "break up" compound forms using the rules in (2) until we get back to p and q.) So L would consist of *valid* propositional forms. An example of a non-valid form would be $\rightarrow ((p$ and this would not be in L.

The elements of L are called *well-formed formulae* (w.f.f.).

Of course, if we have a propositional form with 3 or more variables then we have to enlarge the alphabet E to include the extra variables and change rule (1) to say that all these variables are in L.

*If we look back at all the propositional forms that occurred in the examples of Sections 1 and 2 and just look upon them as words in a language (i.e. forget that p and q are propositional variables) then they are all w.f.f. The question of whether a form was a w.f.f. was a complication that I did not want to waste time with in those sections. Given a form though, we can consider it simply as a word in E^*. The inductive definition of L means that it is possible to write a computer program to see if this word is in the language, L, or not, i.e. we can check if it is a w.f.f. Because the property of a propositional form being a w.f.f. or not does not depend on interpreting p and q etc. as propositional variables we say that being a w.f.f. is a *syntactic* property. Similarly, a deductive proof of validity of a propositional argument does not depend on any interpretation of the symbols p, q, etc. and so is said to be a *syntactic proof*. In a proof of validity using truth tables we need to interpret symbols p, q, etc. as propositional variables, so we have to give a meaning to the words of the argument. For this reason a proof using truth tables is said to be a *semantic proof*. The Completeness Theorem of section 2.4 says, for propositional logic, that valid arguments can be verified by either syntactic or semantic proofs. We should note that semantic proof is often the better of the two types to show that an argument is invalid.