2.3 Rules III

In an argument $P_1, ..., P_n \vdash C$ let $R = P_1 \wedge ... \wedge P_n$. To show the argument is valid we need show that $R \to C$, *or any equivalent propositional form*, is a tautology. In the next rule we consider the case when C is a conditional, that is of the form $p \to q$. So we will try to be proving that $R \to (p \to q)$ is a tautology.

C.P. (Conditional Proof)

By using truth tables it can be checked that we have an equivalence $R \to (p \to q) \equiv (R \wedge p) \to q$. This means that both sides have the same truth values. So to show that $R \to (p \to q)$ is a tautology (i.e. always true). it would suffice to show that $(R \wedge p) \to q$ is a tautology. Yet saying that $(R \wedge p) \to q$ is a tautology is the same as saying that $P_1, ..., P_n, p \vdash q$ is valid. So

To deduce $p \to q$ from premises $P_1, ..., P_n$ it suffices to deduce q from p, $P_1, ..., P_n$.

R.A.A. (Proof by Contradiction)

By using truth tables it can be checked that we have an equivalence $R \to C \equiv (R \wedge (\neg C)) \to O$. So, as above, to show that $R \to C$ is a tautology it would suffice to show that $(R \wedge (\neg C)) \to O$ is a tautology, which is the same as saying that $P_1, ..., P_n, \neg C \vdash O$ is valid. So

To deduce C from $P_1, ..., P_n$ it suffices to deduce a contradiction (O) (usually of the form $s \wedge (\neg s)$ for some proposition s) from $\neg C$, $P_1, ..., P_n$.

D.S. (Disjunctive Syllogism)

The argument $p \vee q, (\neg q) \vdash p$ is valid, (see Ex 19(iii)) so

If steps of the form $p \vee q$ and $\neg q$ occur in the proof, then we can deduce p. Similarly, given steps of the form $p \vee q$ and $\neg p$ we can deduce q.

***Note 1**: An argument with two premises and one conclusion is classically (it goes back to the study of logic in the Middle Ages) known as a *Syllogism*. So, as well as D.S., both MPP and MTT are syllogisms.

Note that the rule D.S. leads to

Example 25 The very simple argument $p,\ \neg\, p\ \vdash\ q$ is valid.

This says that if we start with a premise and its negation (i.e. a contradiction) then we can deduce *anything*.

$$\begin{array}{lll} 1 & p & \text{A} \\ 2 & p \vee q & \vee\text{I } 1 \\ 3 & \neg\, p & \text{A} \\ 4 & q & \text{D.S. } 2{,}3 \end{array}$$

Example 26 (cf. Ex 19(iv)) $p \to q,\ q \to r\ \vdash\ p \to r$

$$\begin{array}{llll} 1 & \lceil & p & \text{A(CP)} \\ 2 & | & p \to q & \text{A} \\ 3 & | & q & \text{MPP } 1{,}2 \\ 4 & | & q \to r & \text{A} \\ 5 & \lfloor & r & \text{MPP } 3{,}4 \\ 6 & & p \to r & \text{CP } 1\text{-}5 \end{array}$$

So $p \to q,\ q \to r\ \vdash\ p \to r$ is a valid argument.

*Note Lines 1-5 give a proof of validity of $p,\ p \to q,\ q \to r\ \vdash\ r$ which we have seen, though with different labels, in Ex 22(i).

Example 27 $A \to B,\ (\neg A) \to B\ \vdash\ B$

$$\begin{array}{llll} 1 & \lceil & \neg B & \text{A(RAA)} \\ 2 & | & A \to B & \text{A} \\ 3 & | & \neg A & \text{MTT } 1{,}2 \\ 4 & | & (\neg A) \to B & \text{A} \\ 5 & | & \neg(\neg A) & \text{MTT } 1{,}4 \\ 6 & | & A & \text{DN } 5 \\ 7 & \lfloor & A \wedge (\neg A) & \wedge\text{I } 3{,}6 \\ 8 & & B & \text{RAA } 1\text{-}7 \end{array}$$

There may be many proofs that an argument is valid. For instance, in this example we also have

$$\begin{array}{llll} 1 & \lceil & \neg B & \text{A(RAA)} \\ 2 & | & A \to B & \text{A} \\ 3 & | & \neg A & \text{MTT } 1{,}2 \\ 4 & | & (\neg A) \to B & \text{A} \\ 5 & | & B & \text{MPP } 3{,}4 \\ 6 & \lfloor & B \wedge (\neg B) & \wedge\text{I } 1{,}6 \\ 7 & & B & \text{RAA } 1\text{-}6 \end{array}$$

So $A \to B,\ (\neg A) \to B\ \vdash\ B$ is a valid argument.

***Note** In line 5 we have deduced B. We cannot then say that the argument is valid. We first have to finish the sub-proof we have started. As stated in line 1 this sub-proof uses proof by contradiction. So we can only finish the sub-proof when we have found a contradiction, as in line 6. The contradiction means that the R.A.A assumption, $\neg B$, cannot hold; that is, we must have B.

2.3.1 Additional Examples

Example 28 (i) Prove that the following argument is valid.

Either Manchester is a city or Stockport is a city.
If Manchester is a city then Manchester has a Cathedral.
Therefore, If Stockport is not a city then Manchester has a Cathedral.

We need to symbolise this, and so we need to break up these compound propositions into the smaller propositions that are common to more than one sentence. So we choose

M = Manchester is a city,
S = Stockport is a city,
C = Manchester has a Cathedral.

The argument then becomes $M \vee S,\ M \to C\ \vdash\ (\neg S) \to C$.

The conclusion here, $(\neg S) \to C$, is a conditional so we think of using Conditional Proof and adding the left hand side of the conditional, i.e. $\neg S$, to our list of premises.

1	$\lceil$	$\neg S$	A(CP)
2	$\vert$	$M \vee S$	A
3	$\vert$	M	DS 1,2
4	$\vert$	$M \to C$	A
5	$\lfloor$	C	MPP 3,4
6		$(\neg S) \to C$	CP 1-5

Thus we see that the argument is valid.

(Note, this argument is valid but the propositions in it are *not* all true! Stockport is not a city.)

(ii) Prove that the following argument is valid.

> If Manchester is a city then Stockport is a city.
> Either Stockport is a city or Manchester has a cathedral.
> If Manchester has a Cathedral then Manchester is a city.
> Therefore, Stockport is a city.

In the notation above this is symbolised as $M \to S,\ S \vee C,\ C \to M \ \vdash\ S$. We can give two proofs for this, neither particularly easy.

1	$\lceil$	$\neg S$	A(RAA)
2	$\vert$	$S \vee C$	A
3	$\vert$	C	D.S. 1,2
4	$\vert$	$C \to M$	A
5	$\vert$	M	MPP 3,4
6	$\vert$	$M \to S$	A
7	$\vert$	$\neg M$	MTT 1,6
8	$\lfloor$	$(\neg M) \wedge M$	$\wedge$I 5,7
9		S	RAA 1-8.

Alternatively,

1		$S \vee C$	A
2	[	S	A($\vee$E)
3	$\lceil$	C	A($\vee$E)
4	$\vert$	$C \to M$	A
5	$\vert$	M	MPP 3,4
6	$\vert$	$M \to S$	A
7	$\lfloor$	S	MPP 5,6
8		S	$\vee$E 1-8.

Note that the first subproof starts and finishes in the same line, line 2.

(iii) Prove that the following argument is valid.

> If Manchester is a city then Stockport is a city.
> Manchester is a city but Stockport is not a city.
> Therefore, Manchester has a Cathedral.

Notice how the premises concern cities, while the conclusion is about a cathedral. We symbolise the argument as $M \to S,\ M \wedge (\neg S) \ \vdash\ C$.

We can see clearly here how the conclusion does not occur in the premises. We could introduce $\neg C$ into the list of premises by using RAA but what would we do with it? Instead, we will look for a contradiction in the premises, since we know from example 25 that we can deduce *anything* from a contradiction.

1	$M \wedge (\neg S)$	A
2	M	$\wedge$E 1
3	$M \rightarrow S$	A
4	S	MPP 2,3
5	$S \vee C$	$\vee$I 4
6	$\neg S$	$\wedge$E 1
7	C	DS 5,6

The contradiction we were looking for is having S (on line 4) and $\neg S$ (on line 6). The C was introduced via DS. Thus the argument is valid.

2.3.2 Harder examples.

Note, RAA and CP can be used within a proof, at any time, not just at the start.

Example 29 $\neg\,(P \wedge (\neg\, Q)) \ \vdash\ P \rightarrow Q$

Idea, use CP and deduce Q from $P, \neg\,(P \wedge (\neg\, Q))$.

To do this use RAA and deduce a contradiction from assuming all of $\neg\, Q$, P, and $\neg\,(P \wedge (\neg\, Q))$.

1	$\neg\,(P \wedge (\neg\, Q))$	A
2	P	A(CP)
3	$\neg\, Q$	A(RAA)
4	$P \wedge (\neg\, Q)$	$\wedge$I 2,3
5	$(P \wedge (\neg\, Q)) \wedge (\neg\,(P \wedge (\neg\, Q)))$	$\wedge$I 1,4
6	Q	RAA 3-5
7	$P \rightarrow Q$	CP 2-6

So $\neg\,(P \wedge (\neg\, Q)) \ \vdash\ P \rightarrow Q$ is a valid argument.

Example 30 $P \rightarrow (Q \wedge R) \ \vdash\ (P \rightarrow Q) \wedge (P \rightarrow R)$

Idea: Deduce $P \rightarrow Q$ using CP.

Deduce $P \rightarrow R$ using CP.

Combine $P \rightarrow Q$ and $P \rightarrow R$ using $\wedge$I.

1		$P \to (Q \wedge R)$	A
2	$\lceil$	P	A (CP)
3	$\vert$	$Q \wedge R$	MPP 1,2
4	$\lfloor$	Q	$\wedge$E 3
5		$P \to Q$	CP 2-4
6	$\lceil$	P	A (CP)
7	$\vert$	$Q \wedge R$	MPP 1,6
8	$\lfloor$	R	$\wedge$E 7
9		$P \to R$	CP 6-8
10		$(P \to Q) \wedge (P \to R)$	$\wedge$I 5,9

So $P \to (Q \wedge R) \vdash (P \to Q) \wedge (P \to R)$ is a valid argument.

Note When a sub-proof is finished you can never refer to lines in the sub-proof again. So, in the third example, when you come to prove $P \to R$, you cannot refer to P in line 2. You need to start again as in line 6.

Subproofs should never intersect each other. You might see

1		$\cdots$		1			$\cdots$
2	$\lceil$	$\cdots$		2	$\lceil$		$\cdots$
3	$\vert$	$\cdots$		3	$\vert$	$\lceil$	$\cdots$
4	$\lfloor$	$\cdots$		4	$\vert$	$\vert$	$\cdots$
5		$\cdots$	or	5	$\vert$	$\vert$	$\cdots$
6	$\lceil$	$\cdots$		6	$\vert$	$\vert$	$\cdots$
7	$\vert$	$\cdots$		7	$\vert$	$\lfloor$	$\cdots$
8	$\lfloor$	$\cdots$		8	$\lfloor$		$\cdots$
9		$\cdots$		9			$\cdots$

but you should never see

1			$\cdots$
2	$\lceil$		$\cdots$
3	$\vert$	$\lceil$	$\cdots$
4	$\vert$	$\vert$	$\cdots$
5	$\vert$	$\vert$	$\cdots$
6	$\lfloor$	$\vert$	$\cdots$
7		$\vert$	$\cdots$
8		$\lfloor$	$\cdots$
9			$\cdots$

Example 31 We can construct new arguments out of old. For instance

(a) In Ex 25 we saw that $p,\ \neg\, p\ \vdash\ q$ is valid.

(b) $p\ \vdash\ (\neg\, p) \rightarrow q$.

We might attempt C.P., which tells us that it suffices to deduce q from p and $\neg\, p$. This takes us back to Example 25 where we showed that $p,\ \neg\, p\ \vdash\ q$ is valid.

The proof will look like

1	⌈	$\neg\, p$	A(CP)
2	|	p	A
3	|	$p \vee q$	$\vee$I 2
4	⌊	q	D.S. 1,3
5		$(\neg\, p) \rightarrow q$	CP 1-3

We could go further, looking at

(c) $\vdash\ p \rightarrow ((\neg\, p) \rightarrow q)$.

There are **no** premises. We have to use a method of proof that allow us to assume new premises, i.e. C.P. or R.A.A.. In this case we again attempt C.P., which tells us that it suffices to deduce $(\neg\, p) \rightarrow q$ from p. Yet this is exactly what we have done in part (b).

The complete proof will look like

1	⌈		p	A(CP)
2	|	⌈	$\neg\, p$	A(CP)
3	|	|	$p \vee q$	$\vee$I 1
4	|	⌊	q	D.S. 2,3
5	⌊		$(\neg\, p) \rightarrow q$	CP 2-4
6			$p \rightarrow ((\neg\, p) \rightarrow q)$	CP 1-5

So $\vdash\ p \rightarrow ((\neg\, p) \rightarrow q)$ is a valid argument.

*Note that $p \rightarrow ((\neg\, p) \rightarrow q)$ is a tautology.

*In general, if $P(p_1, ..., p_n)$ is a tautology then $\vdash P(p_1, ..., p_n)$ is a valid argument.

*2.4 Truth Tables vs. Natural Deduction

At the start of Section 2 we stressed that a proof by natural deduction makes no reference to truth tables. To highlight this separation of proof by truth tables and proof by deduction authors often introduce the symbol $\vDash$. So now, saying that $P_1,\ P_2,\ ...,\ P_r\ \ \vDash\ C$ is valid means $P_1\wedge\ P_2\wedge\ ...\wedge$

$P_r \to C$ is a tautology while saying $P_1,\ P_2,\ \dots\ , P_r\ \ \vdash\ C$ is valid means that C follows from $P_1, P_2, \dots, P_r$ by the rules of inference. We do not make use of the $\vDash$ symbol in this course.

The so-called *Completeness Theorem* for propositional logic states that $P_1,\ P_2,\ \dots,\ P_r\ \ \vDash\ C$ is valid if, and only if, $P_1, P_2, \dots, P_r\ \ \vdash\ C$ is valid. The proof of this result is beyond the scope of this course.

Note: We have given a very long list of rules of inference, and some of them follow from combinations of others, but the more rules we have the shorter the proofs will be. Other authors may well give different lists.