2.2 Rules II

$\wedge$E (Eliminating the 'and')

$p \wedge q \vdash p$ and $p \wedge q \vdash q$ are both valid, so

If we have a step of the form $p \wedge q$ in the proof, then we can deduce p and we can deduce q.

$\wedge$I (Introducing 'and')

$p,\ q \vdash p \wedge q$ is valid, so

If we have steps of the form p and q in the proof, then we can deduce $p \wedge q$.

$\vee$I (Introducing 'or')

$p \vdash p \vee q$ and $p \vdash q \vee p$ are valid, so

If a step of the form p occurs , then we can deduce $p \vee q$, and we can deduce $q \vee p$, for any q.

$\vee$E (Eliminating 'or')

$p \vee q,\ p \to r,\ q \to r \vdash p$ is valid, so

If a step of the form $p \vee q$ occurs and we can deduce r from p and we can deduce r from q, then we can deduce r.

Notes (i) $\vee$I is important as it is the only rule of inference that allows a proposition to be in the conclusion that isn't in the premises.

(ii) In $\vee$E we require two *sub-proofs*, i.e. add p to the original premises and deduce r **then** add q to the original premises and deduce r. We then "sum up" by saying that in both cases we deduce r.

Example 23 (i) $A,\ A \to D \vdash A \wedge D$

1	A	A
2	$A \to D$	A
3	D	MPP 1,2
4	$A \wedge D$	$\wedge$I, 1,3

(ii) $A,\ (A \vee B) \to D \vdash D$

1	A	A
2	$A \vee B$	$\vee$I,1
2	$(A \vee B) \to D$	A
3	D	MPP 1,2

(iii) $A \wedge C,\ (A \vee B) \rightarrow D \ \vdash\ D \wedge C$

1	$A \wedge C$	A
2	A	$\wedge$E 1
3	$A \vee B$	$\vee$I 2
4	$(A \vee B) \rightarrow D$	A
5	D	MPP 3,4
6	C	$\wedge$E 1
7	$D \wedge C$	$\wedge$I 5,6

Therefore the argument is valid.

Example 24 (i) $p \vee q,\ \ p \rightarrow s,\ \ q \rightarrow t \ \vdash\ \ s \vee t$

1		$p \vee q$	A	
2	⌈	p	A($\vee$E)	
3	|	$p \rightarrow s$	A	first sub-proof
4	|	s	MPP 2,3	
5	⌊	$s \vee t$	$\vee$I 4	
6	⌈	q	A($\vee$E)	
7	|	$q \rightarrow t$	A	second sub-proof
8	|	t	MPP 6,7	
9	⌊	$s \vee t$	$\vee$I 8	
10		$s \vee t$	$\vee$E 2-9	

Therefore the argument is valid.

(ii) $p \vee q,\ \ s \rightarrow \neg\, p,\ \ s \rightarrow \neg\, q \ \vdash\ \ \neg\, s$

1		$p \vee q$	A	
2	⌈	p	A($\vee$E)	
3	|	$\neg\ (\neg\, p)$	DN 2	first sub-proof
4	|	$s \rightarrow \neg\, p$	A	
5	⌊	$\neg\, s$	MTT 3,4	
6	⌈	q	A($\vee$E)	
7	|	$\neg\ (\neg\, q)$	DN 6	second sub-proof
8	|	$s \rightarrow \neg\, q$	MPP 6,7	
9	⌊	$\neg\, s$	MTT 7,8	
10		$\neg\, s$	$\vee$E 2-9	

Examples (i) $A \to B,\ A \ \vdash\ B,$ (ii) $(p \vee q) \to B,\ p \vee q \ \vdash\ B,$
(iii) $(p \vee q) \to (s \wedge t)\,,\ p \vee q \ \vdash\ s \wedge t.$
Solutions

(i)

1	A	A
2	$A \to B$	A
3	B	MPP 1,2

(ii)

1	$p \vee q$	A
2	$(p \vee q) \to B$	A
3	B	MPP 1,2

(iii)

1	$p \vee q$	A
2	$(p \vee q) \to (s \wedge t)$	A
3	$s \wedge t$	MPP 1,2

Examples continued (iv) $(m \to n) \to B,\ m \to n \ \vdash\ B,$
(v) $(m \to n) \to (k \to \ell)\,,\ m \to n \ \vdash\ k \to \ell,$
Solutions

(iv)

1	$m \to n$	A
2	$(m \to n) \to B$	A
3	B	MPP 1,2

(v)

1	$m \to n$	A
2	$(m \to n) \to (k \to \ell)$	A
3	$k \to \ell$	MPP 1,2

So all these five arguments have the same form.

Examples (i) $A \to B,\ \neg B \ \vdash\ \neg A,$ (ii) $(\neg p) \to (\neg q)\,,\ q \ \vdash\ p$
(iii) $(m \to n) \to (k \to \ell)\,,\ \neg\,(k \to \ell) \ \vdash\ \neg\,(m \to n)\,.$
Solutions

(i)

1	$A \to B$	A
2	$\neg B$	A
3	$\neg A$	MTT 1,2

(ii)

1	$(\neg p) \to (\neg q)$	A
2	q	A
3	$\neg\,(\neg q)$	DN 2
4	$\neg\,(\neg p)$	MTT 1,3
5	p	DN 4

(iii)

1	$(m \to n) \to (k \to \ell)$	A
2	$\neg\,(k \to \ell)$	A
3	$\neg\,(m \to n)$	MTT 1,2

So these arguments have essentially the same form.

Examples (i) $p,\ q\ \vdash\ p \wedge q$ (ii) $A \vee B,\ B \wedge C\ \vdash\ (A \vee B) \wedge (B \wedge C)$.
Solutions

(i)

1	p		A
2	q		A
3	$p \wedge q$		$\wedge$I 1,2

(ii)

1	$A \vee B$		A
2	$B \wedge C$		A
3	$(A \vee B) \wedge (B \wedge C)$		$\wedge$I 1,2

Examples (i) $p\ \vdash\ p \vee q$ (ii) $A \vee B\ \vdash\ (A \vee B) \vee (B \wedge C)$.
Solutions

(i)

1	p		A
2	$p \vee q$		$\vee$I 1

(ii)

1	$A \vee B$		A
2	$(A \vee B) \vee (B \wedge C)$		$\vee$I 1

Examples (i) $p \vee q,\ p \to r,\ q \to r\ \vdash\ r$

(ii) $(A \vee B) \vee (B \vee C),\ (A \vee B) \to (C \wedge D),\ (B \vee C) \to (C \wedge D)\ \vdash C \wedge D$.

Solutions

(i)

1		$p \vee q$	A
2	⌈	p	A($\vee E$)
3	|	$p \to r$	A
4	⌊	r	MPP 2,3
5	⌈	q	A($\vee E$)
6	|	$q \to r$	A
7	⌊	r	MPP 5,6
8		r	$\vee$E 2-7

(ii)

1		$(A \vee B) \vee (B \vee C)$	A
2	⌈	$A \vee B$	$\vee$E 1
3	|	$(A \vee B) \to (C \wedge D)$	A
4	⌊	$C \wedge D$	MPP 2,3
5	⌈	$B \vee C$	$\vee$E 1
6	|	$(B \vee C) \to (C \wedge D)$	A
7	⌊	$C \wedge D$	MPP 5,6
8		$C \wedge D$	$\vee$E 2-7

Examples (i) $p \vee q,\ p \to r,\ \neg r \to \neg q \ \vdash\ r$
(ii) $(m \wedge n) \vee (s \vee t)\,,\ (m \wedge n) \to (k \wedge \ell)\,,\ (\neg\,(k \wedge \ell)) \to (\neg\,(s \vee t)) \ \vdash\ k \wedge \ell.$
Solutions

(i)

1		$p \vee q$	A
2	⌈	p	A($\vee E$)
3	|	$p \to r$	A
4	⌊	r	MPP 2,3
5	⌈	q	A($\vee E$)
6	|	$\neg\,(\neg q)$	DN 5
7	|	$\neg r \to \neg q$	A
8	|	$\neg\,(\neg r)$	MTT 6,7
9	⌊	r	DN 8
10		r	$\vee E$ 2-9

(ii)

1		$(m \wedge n) \vee (s \vee t)$	A
2	⌈	$m \wedge n$	A($\vee E$)
3	|	$(m \wedge n) \to (k \wedge \ell)$	A
4	⌊	$k \wedge \ell$	MPP 2,3
5	⌈	$s \vee t$	A($\vee E$)
6	|	$\neg\,(\neg\,(s \vee t))$	DN 5
7	|	$(\neg\,(k \wedge \ell)) \to (\neg\,(s \vee t))$	A
8	|	$\neg\,(\neg\,(k \wedge \ell))$	MTT 6,7
9	⌊	$k \wedge \ell$	DN 8
10		$k \wedge \ell$	$\vee E$ 2-9

Examples (i) $p,\ p \to q,\ q \to r \ \vdash\ r$
(ii) $s \vee t,\ \ (s \vee t) \to (m \wedge n)\,,\ \ (m \wedge n) \to (k \to \ell) \ \vdash\ k \to \ell$
Solutions

(i)

1	p		A
2	$p \to q$		A
3	q		MPP 1,2
4	$q \to r$		A
5	r		MPP 3,4

(ii)

1	$s \vee t$		A
2	$(s \vee t) \to (m \wedge n)$		A
3	$m \wedge n$		MPP 1,2
4	$(m \wedge n) \to (k \to \ell)$		A
5	$k \to \ell$		MPP 3,4

Examples (i) $p,\ q \to \neg p,\ \ (\neg r) \to q \ \vdash\ r$
(ii) $A \to B,\ \ (C \to D) \to \neg\,(A \to B)\,,\ \ (\neg\,(E \to F)) \to (C \to D) \ \vdash\ E \to F$
Solutions

(i)

1	p		A
2	$q \to \neg p$		A
3	$\neg\,(\neg p)$		DN 1
4	$\neg q$		MTT 2,3
5	$(\neg r) \to q$		A
6	$\neg\,(\neg r)$		MTT 4,5
7	r		DN 6

(ii)

1	$A \to B$		A
2	$(C \to D) \to \neg\,(A \to B)$		A
3	$\neg\,(\neg\,(A \to B))$		DN 1
4	$\neg\,(C \to D)$		MTT 2,3
5	$(\neg\,(E \to F)) \to (C \to D)$		A
6	$\neg\,(\neg\,(E \to F))$		MTT 4,5
7	$E \to F$		DN 6