1.3 More Connectives

1.3.1 Conditional and Biconditional

Example 12

Consider the proposition: "If it is raining then it is cloudy", which we say is a *conditional* statement.

Let $p =$"It is raining", $q =$"It is cloudy". Then the proposition can be written as "If p then q".

We symbolize this as $p \to q$. This can also be read as "p *implies* q".

The truth table for $\to$ is

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The important case is the second one. We never want something false to follow from something true; i.e. we do **not** want "If p then q" to be true if p is true and q is false. This is the second line above.

*If p is false in a $p \to q$ statement then we have to give some truth-value to $p \to q$. It is a **convention** to assign the value of True, whatever the truth-value of q. This leads to the last two lines in the table.

Note

Consider "p only if q". With the choice of p and q above this becomes "It is raining only if it is cloudy". If you believe this statement and you look out of the window to see it is raining you will conclude that it must be cloudy. That is, raining has implied cloudy, (and apart from occasional freak weather conditions, this is what we would expect). Thus we can rewrite the sentence as "If it is raining then it is cloudy"; i.e. "If p then q" or $p \to q$.

Now consider "p if q". Again, with p and q as above, this becomes "It is raining if it is cloudy". This is telling us that every time it is cloudy then it must be raining (not a proposition we would believe). Thus we can rewrite the sentence as "If it is cloudy then it is raining"; i.e. "If q then p" or $q \to p$.

So we symbolize "p only if q" as $p \to q$
and symbolize "p if q" as $q \to p$.

Definition

The *biconditional* $p \leftrightarrow q$ is defined by

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

.

So $p \leftrightarrow q$ is true if p and q have the same truth-values, false if p and q have different truth-values.

Example 13

Note

p	q	$q \to p$	$p \to q$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	T	F	F
F	T	F	T	F
F	F	T	T	T

.

So

$$\begin{aligned} p \leftrightarrow q &\equiv (q \to p) \wedge (p \to q) \\ &\equiv (p \text{ if } q) \text{ and } (p \text{ only if } q) && \text{(by notes above)} \\ &\equiv p \text{ if, and only if, } q \end{aligned}$$

So we read $p \leftrightarrow q$ as "p if, and only if, q" or, in shorthand, "p iff q".

***Example 14**

Let $P(p_1, ..., p_n)$ and $Q(p_1, ..., p_n)$ be two propositional forms (with variables $p_1, ..., p_n$). If

$$P(p_1, ..., p_n) \equiv Q(p_1, ..., p_n)$$

then P and Q are equivalent so

P and Q always take the same truth-value,
i.e. $P \leftrightarrow Q$ is always true,
i.e. $P \leftrightarrow Q$ is a tautology.

So $P \equiv Q$ is the same as "$P \leftrightarrow Q$ is a tautology".

More Rules

Using truth tables the student should check the equivalences in the following three examples:

Example 15 $p \to q \equiv (\neg q) \to (\neg p)$.

We say that $(\neg q) \to (\neg p)$ is the *contrapositive* of $p \to q$.

So the contrapositive of "If it is raining then it is cloudy" is "If it is not cloudy then it is not raining".

Note

We say that $q \to p$ is the *converse* of $p \to q$. So the converse of "If it is raining then it is cloudy" is "If it is cloudy then it is raining".

Important Converse and contrapositive are different ideas. If a conditional proposition is true then it's contrapositive will be true though it's converse might well be false.

Example 16 $\neg (p \to q) \equiv p \wedge (\neg q)$.

So "It is not the case that if it is raining then it is cloudy" is equivalent to "It is raining and it is not cloudy".

Example 17 $p \to q \equiv (\neg p) \vee q$.

So a propositional form containing $\to$ can be written without the $\to$. Similarly

$$
\begin{aligned}
p \leftrightarrow q &\equiv (p \to q) \wedge (q \to p) && \text{(by Ex13)} \\
&\equiv ((\neg p) \vee q) \wedge ((\neg q) \vee p). && \text{(by Ex17)}
\end{aligned}
$$

So propositional forms containing $\leftrightarrow$ can be written without the $\leftrightarrow$.

Example 18 Rewrite $(p \to q) \to q$ without the $\to$.

Solution.

$$
\begin{aligned}
(p \to q) \to q &\equiv ((\neg (p \to q)) \vee q) && \text{(by Ex17)} \\
&\equiv ((p \wedge (\neg q)) \vee q) && \text{(by Ex16)} \\
&\equiv ((p \vee q) \wedge ((\neg q) \vee q)) && \text{(Distributive law)} \\
&\equiv (p \vee q) \wedge I && \text{(5a)} \\
&\equiv p \vee q. && \text{(4b)}
\end{aligned}
$$