

Solutions to Additional Questions 4 and 5

4 Exercises in C.P.

1. $q \rightarrow r, p, p \rightarrow q \vdash r,$

1	p	A
2	$p \rightarrow q$	A
3	q	MPP 1,2
4	$q \rightarrow r$	A
5	r	MPP 3,4

2. $q \rightarrow r, p \rightarrow q \vdash p \rightarrow r,$

1	[p	A(CP)
2		$p \rightarrow q$	A
3		q	MPP 1,2
4		$q \rightarrow r$	A
5		r	MPP 3,4
6]	$p \rightarrow r$	CP 1-5

Note that lines 1-5 are simply the proof for the argument in Question 1.

3. $p \rightarrow q \vdash (q \rightarrow r) \rightarrow (p \rightarrow r),$

1	[$q \rightarrow r$	A(CP)	
2		[p	A(CP)
3			$p \rightarrow q$	A
4			q	MPP 2,3
5			r	MPP 1,4
6]	$p \rightarrow r$	CP 2-5
7]	$(q \rightarrow r) \rightarrow (p \rightarrow r)$	CP 1-6	

Note that lines 2-6 are simply the proof for the argument in Question 2.

4. $q \rightarrow r, p \vdash (p \rightarrow q) \rightarrow r,$

1	[$p \rightarrow q$	A(CP)
2		p	A
3		q	MPP 1,2
4		$q \rightarrow r$	A
5		r	MPP 3,4
6]	$(p \rightarrow q) \rightarrow r$	CP 1-5

5. $q \rightarrow r \vdash p \rightarrow ((p \rightarrow q) \rightarrow r)$,

1	[p	A(CP)
2		[$p \rightarrow q$	A(CP)
3		q	MPP 1,2
4		$q \rightarrow r$	A
5		r	MPP 3,4
6]	$(p \rightarrow q) \rightarrow r$ CP 2-5
7]	$p \rightarrow ((p \rightarrow q) \rightarrow r)$	CP 1-5

6. $q \rightarrow r \vdash (p \wedge (p \rightarrow q)) \rightarrow r$,

1	[$p \wedge (p \rightarrow q)$	A(CP)
2		$p \rightarrow q$	$\wedge E$ 1
3		p	$\wedge E$ 1
4		q	MPP 2,3
5		$q \rightarrow r$	A
6		r	MPP 4,5
7]	$(p \wedge (p \rightarrow q)) \rightarrow r$	CP 1-6

5 Arguments with no premises

Now you definitely need to use rules that increase the number of premises

1. $\vdash (p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$.

Hint: Qu 4.3

1	[$p \rightarrow q$	A(CP)
2		[$q \rightarrow r$	A(CP)
3		[p	A(CP)
4		q	MPP 1,3
5		r	MPP 2,4
6] $p \rightarrow r$	CP 3-5
7] $(q \rightarrow r) \rightarrow (p \rightarrow r)$	CP 2-6
8]	$(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$	CP 1-7

$$2. \vdash (q \rightarrow r) \rightarrow (p \rightarrow ((p \rightarrow q) \rightarrow r)).$$

Hint: Qu 4.5

1	[$q \rightarrow r$	A(CP)		
2		[p	A(CP)	
3			[$p \rightarrow q$	A(CP)
4				q	MPP 2,3
5				r	MPP 1,4
6]	$(p \rightarrow q) \rightarrow r$	CP 3-5
7]	$p \rightarrow ((p \rightarrow q) \rightarrow r)$	CP 2-6	
8]		$(q \rightarrow r) \rightarrow (p \rightarrow ((p \rightarrow q) \rightarrow r))$	CP 1-7	

$$3. \vdash ((q \rightarrow r) \wedge p \wedge (p \rightarrow q)) \rightarrow r.$$

1	[$((q \rightarrow r) \wedge p \wedge (p \rightarrow q))$	A(CP)
2		$q \rightarrow r$	$\wedge E$ 1
3		p	$\wedge E$ 1
4		$p \rightarrow q$	$\wedge E$ 1
5		q	MPP 3,4
6		r	MPP 2,5
7]	$((q \rightarrow r) \wedge p \wedge (p \rightarrow q)) \rightarrow r$	CP 1-6

$$4. \vdash p \vee (\neg p).$$

Hint: Use *RAA* and the method of proofs of Questions 3.3 and 3.4

1	[$\neg(p \vee (\neg p))$	A(RAA)	
2		[p	A(RAA)
3			$p \vee (\neg p)$	$\vee I$ 2
4			$(p \vee (\neg p)) \wedge (\neg(p \vee (\neg p)))$	$\wedge I$ 1,3
5			$\neg p$	RAA 2-4
6		[$\neg p$	A(RAA)
7			$p \vee (\neg p)$	$\vee I$ 6
8			$(p \vee (\neg p)) \wedge (\neg(p \vee (\neg p)))$	$\wedge I$ 1,7
9			$\neg(\neg p)$	RAA 6-8
10			p	DN 9
11			$p \wedge (\neg p)$	$\wedge I$ 5,10
12			$p \vee (\neg p)$	RAA 1-11

This is a surprisingly long proof for a simple result. An occasion where using truth tables would be quicker.

I leave it to the student to check, using truth tables perhaps, that the conclusions of the above arguments are all tautologies. In fact, a propositional form P is a tautology if, and only if, $\vdash P$ is a valid argument.

$$5. \vdash (s \rightarrow t) \vee (s \wedge (\neg t)).$$

I don't give the proof here, but note that if we write p for $s \rightarrow t$ then $s \wedge (\neg t)$ is $\neg p$, so the "form" of the argument is identical to that seen in Question 4.