Solutions to Additional Questions 2 and 3

2 Arguments with only one premise.

These can be a little tricky to prove. You often need to use a rule that increases the numbers of premises, i.e. CP or RAA.

1. $A \to \neg A \vdash \neg A$.

1	⌈	$\neg(\neg A)$	A(RAA)
2	|	A	DN 1
3	|	$A \to \neg A$	A
4	|	$\neg A$	MPP 2,3
5	⌊	$A \wedge (\neg A)$	∧I 2,4
6		$\neg A$	RAA 1-5

2. $\neg p \vdash p \to q$.

1	⌈	p	A(CP)
2	|	$p \vee q$	∨I 1
3	|	$\neg p$	A
4	⌊	q	DS 2,3
5		$p \to q$	CP 1-4

3. $A \to T \vdash (\neg T) \to (\neg A)$,

1	⌈	$\neg T$	A(CP)
2	|	$A \to T$	A
3	⌊	$\neg A$	MTT 1,2
4		$(\neg T) \to (\neg A)$	CP 1-3

(Why can we immediately deduce that $(\neg T) \to (\neg A) \vdash A \to T$ is valid? Relabel the propositions, i.e. relabel A as $\neg T$ and T as $\neg A$.)

4. $p \to q \vdash (p \wedge r) \to (q \wedge r)$.

1	$p \wedge r$	A(CP)
2	p	∧E 1
3	$p \to q$	A
4	q	MPP 2,3
5	r	∧E 1
6	$q \wedge r$	∧I 4,5

5. $p \to q \vdash (p \lor r) \to (q \lor r)$.

1		$p \lor r$	A(CP)
2	⌈	p	$\lor$E 1
3	|	$p \to q$	A
4	|	q	MPP 2,3
5	⌊	$q \lor r$	$\lor$I 4
6	⌈	r	$\lor$E 1
7	⌊	$q \lor r$	$\lor$I, 6
8		$q \lor r$	$\lor$E 2-7

3 Boolean Laws of Logic.

There is a general principle that if we have a logical equivalence $P \equiv Q$ then we should be able to prove that both $P \vdash Q$ and $Q \vdash P$ are valid using the rules of inference. If both $P \vdash Q$ and $Q \vdash P$ are valid we combine them by writing $P \dashv\vdash Q$. So $P \equiv Q$ if, and only if, $P \dashv\vdash Q$.

Try this out in the following arguments, proving they are all valid.

1 $p \land (q \lor r) \vdash (p \land q) \lor (p \land r)$,

1		$p \land (q \lor r)$	A
2		p	$\land$E 1
3		$q \lor r$	$\land$E 1
4	⌈	q	$\lor$E 3
5	|	$p \land q$	$\land$I 2,4
6	⌊	$(p \land q) \lor (p \land r)$	$\lor$I 5
7	⌈	r	$\lor$E 3
8	|	$p \land r$	$\land$I 2,7
9	⌊	$(p \land q) \lor (p \land r)$	$\lor$I 8
10		$(p \land q) \lor (p \land r)$	$\lor$E 4-9

2 $(p \wedge q) \vee (p \wedge r) \ \vdash \ p \wedge (q \vee r)$,

1		$(p \wedge q) \vee (p \wedge r)$	A
2	$\lceil$	$p \wedge q$	$\vee$E 1
3	$\vert$	p	$\wedge$E 2
4	$\vert$	q	$\wedge$E 2
5	$\vert$	$q \vee r$	$\vee$I 4
6	$\lfloor$	$p \wedge (q \vee r)$	$\wedge$I 3,5
7	$\lceil$	$p \wedge r$	$\vee$E 1
8	$\vert$	p	$\wedge$E 7
9	$\vert$	r	$\wedge$E 7
10	$\vert$	$q \vee r$	$\vee$I 9
11	$\lfloor$	$p \wedge (q \vee r)$	$\wedge$I 8,10
12		$p \wedge (q \vee r)$	$\vee$E 2-11

Hence $p \wedge (q \vee r) \ \dashv\vdash \ (p \wedge q) \vee (p \wedge r)$.

3 $\neg (p \vee q) \ \vdash \ \neg p$,

1	$\lceil$	$\neg (\neg p)$	A(RAA)
2	$\vert$	p	DN 1
3	$\vert$	$p \vee q$	$\vee$I 2
4	$\vert$	$\neg (p \vee q)$	A
5	$\lfloor$	$(p \vee q) \wedge (\neg (p \vee q))$	$\wedge$I 3,4
6		$\neg p$	

4 $\neg (p \vee q) \ \vdash \ \neg q$, similar to the last one.

5 * Combine last two cases to get $\neg (p \vee q) \ \vdash \ (\neg p) \wedge (\neg q)$.

Hint: Use *RAA* twice

1	$\lceil$	$\neg (\neg p)$	A(RAA)
2	$\vert$	p	DN 1
3	$\vert$	$p \vee q$	$\vee$I 2
4	$\vert$	$\neg (p \vee q)$	A
5	$\lfloor$	$(p \vee q) \wedge (\neg (p \vee q))$	$\wedge$I 3,4
6		$\neg p$	RAA 1-5
7	$\lceil$	$\neg (\neg q)$	A(RAA)
8	$\vert$	q	DN 7
9	$\vert$	$p \vee q$	$\vee$I 8
10	$\vert$	$\neg (p \vee q)$	A
11	$\lfloor$	$(p \vee q) \wedge (\neg (p \vee q))$	$\wedge$I 9,10
12		$\neg q$	RAA 7-11
13		$(\neg p) \wedge (\neg q)$	$\wedge$I 6,12

6 $(\neg p) \wedge (\neg q) \ \vdash \ \neg (p \vee q)$.

1	⌈		$\neg(\neg(p \vee q))$	A(RAA)
2	|		$p \vee q$	DN 1
3	|		$(\neg p) \wedge (\neg q)$	A
4	|	⌈	p	$\vee$E 2
5	|	|	$\neg p$	$\wedge$E 3
6	|	⌊	$p \wedge (\neg p)$	$\wedge$I 4,5
7	|	⌈	q	$\vee$E 2
8	|	|	$\neg q$	$\wedge$E 3
9	|	⌊	$q \wedge (\neg q)$	$\wedge$I 7,8
10	⌊		O	$\vee$E 4-9
11			$\neg(p \vee q)$	RAA 1-10

Note, how both p and q led to contradictions in the two subproofs. But these were different contradictions so I summed this up in line 10 by writing O, for contradiction. Of course, from a contradiction you can deduce anything so it would be possible to rewrite the proof so that both subproofs finish with the same contradiction. E.g.

1	⌈		$\neg(\neg(p \vee q))$	A(RAA)
2	|		$p \vee q$	DN 1
3	|		$(\neg p) \wedge (\neg q)$	A
4	|	⌈	p	$\vee$E 2
5	|	|	$\neg p$	$\wedge$E 3
6	|	⌊	$p \wedge (\neg p)$	$\wedge$I 4,5
7	|	⌈	q	$\vee$E 2
8	|	|	$q \vee p$	$\vee$I,7
9	|	|	$\neg q$	$\wedge$E 3
10	|	|	p	D.S. 8,9
11	|	|	$q \vee (\neg p)$	$\vee$I,7
12	|	|	$\neg p$	D.S. 9,11
13	|	⌊	$p \wedge (\neg p)$	$\wedge$I 10,12
14	⌊		$p \wedge (\neg p)$	$\vee$E 4-13
15			$\neg(p \vee q)$	RAA 1-14

7 $\neg p \ \vdash \ p \rightarrow q$,

1	⌈	p	A(CP)
2	|	$p \vee q$	$\vee$I 1
3	|	$\neg p$	A
4	⌊	q	DS 2,3
5		$p \rightarrow q$	CP 1-4

8 $q \vdash p \to q$.

1	$\lceil$	p	A(CP)
2	$\lfloor$	q	A
3		$p \to q$	CP 1-2

9 * Combine the last two to get $\neg(p \to q) \vdash p \wedge (\neg q)$. Hint: Use *RAA* twice

1	$\lceil$		$\neg p$	A(RAA)
2	$\vert$	$\lceil$	p	A(CP)
3	$\vert$	$\vert$	$p \vee q$	$\vee$I 2
4	$\vert$	$\lfloor$	q	DS 1,3
5	$\vert$		$p \to q$	CP 2-4
6	$\vert$		$\neg(p \to q)$	A
7	$\lfloor$		$(p \to q) \wedge (\neg(p \to q))$	$\wedge$I 6,7
8			p	RAA 1-7
9	$\lceil$		$\neg(\neg q)$	A(RAA)
10	$\vert$	$\lceil$	p	A(CP)
11	$\vert$	$\lfloor$	q	DN 9
12	$\vert$		$p \to q$	CP 10-11
13	$\vert$		$\neg(p \to q)$	A
14	$\lfloor$		$(p \to q) \wedge (\neg(p \to q))$	$\wedge$I 12,13
15			$\neg q$	RAA 9-14
16			$p \wedge (\neg q)$	

10 $p \wedge (\neg q) \vdash \neg(p \to q)$.

1	$\lceil$	$\neg(\neg(p \to q))$	A(RAA)
2	$\vert$	$p \to q$	DN 1
3	$\vert$	$p \wedge (\neg q)$	A
4	$\vert$	p	$\wedge$E 3
5	$\vert$	q	MPP 2,4
6	$\vert$	$\neg q$	$\wedge$E 3
7	$\lfloor$	$q \wedge (\neg q)$	$\wedge$I 5,6
8		$\neg(p \to q)$	RAA 1-7

11 $(\neg p) \vee q \vdash p \to q$.

1	$\lceil$	p	A(CP)
2	$\vert$	$\neg(\neg p)$	DN 1
3	$\vert$	$(\neg p) \vee q$	A
4	$\lfloor$	q	DS 2,3
5		$p \to q$	CP 1-4

Why can we immediately deduce that $(\neg q) \vee p \ \vdash \ q \to p$ is valid? Simply relabel the propositions, i.e. replace p by q and q by p.

12 *Combine the two results of Qu. 11 to get

$$(\neg p) \vee q, \ \ (\neg q) \vee p \ \vdash \ \ (p \to q) \wedge (q \to p)$$

which can be written as $(\neg p) \vee q, \ \ (\neg q) \vee p \ \vdash \ p \leftrightarrow q$. Hint: Use CP twice

1	$\lceil$	p	A(CP)
2	$\mid$	$\neg(\neg p)$	DN 1
3	$\mid$	$(\neg p) \vee q$	A
4	$\lfloor$	q	DS 2,3
5		$p \to q$	CP 1-4
6	$\lceil$	q	A(CP)
7	$\mid$	$\neg(\neg q)$	DN 6
8	$\mid$	$(\neg q) \vee p$	A
9	$\lfloor$	p	DS 7,8
10		$q \to p$	CP 6-9
11		$(p \to q) \wedge (q \to p)$	$\wedge$I 5,10

13 $\neg(p \wedge q) \ \vdash \ p \to (\neg q)$.

1	$\lceil$		p	A(CP)
2	$\mid$	$\lceil$	$\neg(\neg q)$	A(RAA)
3	$\mid$	$\mid$	q	DN 2
4	$\mid$	$\mid$	$p \wedge q$	$\wedge$I 1,3
5	$\mid$	$\mid$	$\neg(p \wedge q)$	A
6	$\mid$	$\lfloor$	$(p \wedge q) \wedge (\neg(p \wedge q))$	$\wedge$I 4,5
7	$\lfloor$		$\neg q$	RAA 2-6
8			$p \to (\neg q)$	CP 1-7