Additional Questions

Those marked with a * may be considered harder than the rest.

1 Valid Arguments

Prove the following arguments are valid:

1. $(A \wedge B) \vee C, \ \neg (A \wedge B) \ \vdash \ C,$
2. $(D \to E) \to (F \to G), \ D \to E \ \vdash \ F \to G,$
3. $m \to (p \vee q), \ p \to q, \ m \ \vdash \ q,$
4. $m \to (p \vee q), \ p \to r, \ \ \neg r, \ m \ \vdash \ q,$
5. $m \to (p \vee q), \ p \to r \ , \ \neg r \ \vdash \ m \to q,$
6. Give a proof using RAA, **not** MPP, of $q, \ q \to s \ \vdash \ s,$
7. Give a proof using RAA, **not** MTT, of $q \to s, \ \ \neg s \ \vdash \ \ \neg q,$
8. Use RAA to prove the following:
   8.1. $(\neg p) \to t, \ \ \neg (s \vee t) \ \vdash \ p,$
   8.2. $(\neg s) \to p, \ \ \neg (s \vee t) \ \vdash \ p,$
   8.3. $p \vee s, \ \ \neg (s \vee t) \ \vdash \ p.$
9. Use C.P. to prove $a \to (b \vee r), \ \ \neg r \ \vdash \ a \to b.$
10. * Prove $a \to (b \vee r), \ \ (a \to b) \to c, \ \ \neg (c \vee r) \ \vdash \ r.$
    (Try to reuse the proof of Qu. 9)
11. $G \vee H, \ \ (\neg H) \vee I, \ \ (\neg G) \vee J \ \vdash \ I \vee J.$

2 Arguments with only one premise.

These can be a little tricky to prove. You often need to use a rule that increases the numbers of premises, i.e. CP or RAA.

1. $A \to \ \neg A \ \vdash \ \ \neg A.$
2. $\neg p \ \vdash \ p \to q.$
3. $A \to T \ \vdash \ \ (\neg T) \to (\neg A),$

(Why can we immediately deduce that $(\neg T) \to (\neg A) \ \vdash \ A \to T$ is valid?)

4. $p \to q \ \vdash \ \ (p \wedge r) \to (q \wedge r).$
5. $p \to q \ \vdash \ \ (p \vee r) \to (q \vee r).$

3 Boolean Laws of Logic.

There is a general principle that if we have a logical equivalence $P \equiv Q$ then we should be able to prove that both $P \ \vdash \ Q$ and $Q \ \vdash \ P$ are valid using the rules of inference. If both $P \ \vdash \ Q$ and $Q \ \vdash \ P$ are valid we combine them by writing $P \ \dashv\vdash \ Q$. So $P \equiv Q$ if, and only if, $P \ \dashv\vdash \ Q$.

Try this out in the following arguments, proving they are all valid.

1. $p \wedge (q \vee r) \ \vdash \ (p \wedge q) \vee (p \wedge r)$,
2. $(p \wedge q) \vee (p \wedge r) \ \vdash \ p \wedge (q \vee r)$,
3. $\neg (p \vee q) \ \vdash \ \neg p$,
4. $\neg (p \vee q) \ \vdash \ \neg q$,
5. * Combine last two cases to get $\neg (p \vee q) \ \vdash \ (\neg p) \wedge (\neg q)$.

Hint: Use RAA twice

6. $(\neg p) \wedge (\neg q) \ \vdash \ \neg (p \vee q)$.
7. $\neg p \ \vdash \ p \rightarrow q$,
8. $q \ \vdash \ p \rightarrow q$.
9. * Combine the last two to get $\neg (p \rightarrow q) \ \vdash \ p \wedge (\neg q)$.

Hint: Use RAA twice

10. $p \wedge (\neg q) \ \vdash \ \neg (p \rightarrow q)$.
11. $(\neg p) \vee q \ \vdash \ p \rightarrow q$.

Why can we immediately deduce that $(\neg q) \vee p \ \vdash \ q \rightarrow p$ is valid?

12. *Combine the two results of Qu. 11 to get

$$(\neg p) \vee q, \ (\neg q) \vee p \ \vdash \ (p \rightarrow q) \wedge (q \rightarrow p)$$

which can be written as $(\neg p) \vee q, \ (\neg q) \vee p \ \vdash \ p \leftrightarrow q$.

Hint: Use CP twice

13. $\neg (p \wedge q) \ \vdash \ p \rightarrow (\neg q)$.

4 Exercises in C.P.

1. $q \rightarrow r, \ p, \ p \rightarrow q \ \vdash \ r$,
2. $q \rightarrow r, \ p \rightarrow q \ \vdash \ p \rightarrow r$,
3. $p \rightarrow q \ \vdash \ (q \rightarrow r) \rightarrow (p \rightarrow r)$,
4. $q \rightarrow r, \ p \ \vdash \ (p \rightarrow q) \rightarrow r$,
5. $q \rightarrow r \ \vdash \ p \rightarrow ((p \rightarrow q) \rightarrow r)$,
6. $q \rightarrow r \ \vdash \ (p \wedge (p \rightarrow q)) \rightarrow r$,

5 Arguments with no premises

Now you definitely need to use rules that increase the number of premises

1. $\vdash \ (p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$.

Hint: Qu 4.3

2. $\vdash \ (q \rightarrow r) \rightarrow (p \rightarrow ((p \rightarrow q) \rightarrow r))$.

Hint: Qu 4.5

3. $\vdash\ ((q \to r) \wedge p \wedge (p \to q)) \to r.$
4. $\vdash\ p \vee (\neg p)\,.$

Hint: Use RAA and the method of proof of Qu. 3.5

5. $\vdash\ (s \to t) \vee (s \wedge (\neg t))\,.$

I leave it to the student to check, using truth tables perhaps, that the conclusions of the above arguments are all tautologies. In fact, a propositional form P is a tautology if, and only if, $\vdash\ P$ is a valid argument.

6 Arguments in words.

Symbolise the following arguments and prove them valid using the rules of deduction

1. Adam was the first man and Eve the first woman. If the bible is wrong then Adam wasn't the first man. If the bible is right then Eve wasn't the first woman. Therefore, Noah built the Arc.

(Is it also valid to conclude that Noah didn't build the arc?)

2. Either the fridge is plugged in or I can see the milk. If the fridge door is open and the fridge is plugged in then the fridge light is on. If the fridge light is off then I cannot see the milk. Therefore, if the fridge door is open then the fridge light is on.

3. Annie cycles to work and Steph walks to the shops. If Annie cycles to work then either Cynthia or Ron walk to the shops, If both Cynthia and Steph walk to the shops then Ron also walks to the shops. Therefore, Ron walks to the shops.

4. If $2+3 \neq 5$ then either $2 \times 3 = 7$ or $2 \times 3 = 5$. But $2+3=5$ only if $6=7$. Yet $6 \neq 7$ and $2 \times 3 \neq 7$. Hence $2 \times 3 = 5$.

5. If either Buxton or Macclesfield lie on the A6 then Stockport is not on the A6. If both Stockport and Longsight lie on the A6 then Buxton is on the A6. Therefore, if Stockport lies on the A6 then Longsight is not on the A6.

Hint: CP followed by RAA

7 Invalid Arguments.

Do not draw up truth tables for the following arguments but, instead, find truth values for the variables in each case that make all the premises true and conclusion false. In this way show that each of the following arguments is invalid.

1. $A \to B,\ B \vdash A,$
2. $C \to D,\ \neg C \vdash \neg D,$
3. $E \vee G,\ G \vee H \vdash E \vee H,$
4. $(I \vee J) \to (K \wedge L)\,,\ K \wedge L \vdash I \vee J,$
5. $(m \vee n) \wedge (m \vee p)\,,\ (n \vee q) \wedge (p \vee q) \vdash m \vee q,$
6. $r \to (s \wedge t)\,,\ s \wedge (\neg u) \vdash r \leftrightarrow u,$
7. $v \to (w \vee x)\,,\ y \to (\neg w)\,,\ (\neg z) \vee y,\ x \to (a \wedge (\neg z))\,,\ z \to a \vdash b \to (\neg z)\,.$

8 Sets

1) (i) By looking at Venn diagrams, find which if any of the following are true:

$$\begin{aligned} A \cap (B \bigtriangleup C) &= (A \cap B) \bigtriangleup (A \cap C)\,, \\ A \bigtriangleup (B \cap C) &= (A \bigtriangleup B) \cap (A \bigtriangleup C)\,. \end{aligned}$$

(ii) Starting from

$$D \bigtriangleup E = (D \cup E) \cap (D \cap E)^c\,,$$

use the Boolean laws of sets to show

$$D \bigtriangleup E = (D \cap E^c) \cup (D^c \cap E)\,.$$

(iii) Use (ii) along with the Boolean laws of sets to prove the valid equality in (i).

Compare this question to Sheet 2 Question 9.

9 Cardinalities of Sets

1) A number of people were asked about their reading habits of national papers. The results were as follows.

- All readers of the Times read the Sun.
- Every person either reads the Sun or doesn't read the Mirror.
- 11 people read the Sun but don't read the Mirror.
- 8 people read either the Times or the Mirror but not both.

- 10 people read the Sun and either read the Mirror or do not read the Times.
- 14 people either read the Sun and not the Mirror or read both the Sun and Times.
- 9 people read neither the Times nor the Mirror.

Calculate how many people read each paper and how many people were surveyed.

2) About three sets $A, B, C \subseteq U$ you are told that

$$\begin{array}{ll} |U| = 27, & |A \triangle (B \cap C)| = 12, \\ |(A \cap B) \triangle (B \cap C)| = 5, & |(B \cup C)^c| = 5, \\ |C \triangle (B \triangle A)| = 16, & |C \setminus (B \setminus A)| = 12, \\ |A \cap (B \triangle C)| = 8, & |C \cap (A \cup B)| = 7. \end{array}$$

Find $|A|, |B|$ and $|C|$.

Hints

1.10 First use RAA. Then try to use the proof of Question 1.9. The proof in Question 1.9 derives $a \to b$, obviously of use here because of the second premise.

2.3 When you have $A \to T \ \vdash \ (\neg T) \to (\neg A)$, the T and A are simply labels and can be relabelled. In stages, we can replace A by $\neg\, p$ and T by $\neg\, q$ to get $(\neg p) \to (\neg q) \ \vdash \ q \to p$. A further relabelling of p to T and q to A gives $(\neg T) \to (\neg A) \ \vdash \ A \to T$, as required.

3.5 Simply use RAA twice; do not be worried about starting RAA part way through the proof. The arguments from 3.3 and 3.4 will give $\neg\, p$ and $\neg\, q$ respectively, which can then be combined by $\wedge I$.

3.9 Use RAA twice. So, for example, for the first time assume $\neg\, p$. The argument from 3.7 will give you $p \to q$ which contradicts the premise of the present argument.

3.11 $(\neg\, q) \vee p \ \vdash \ q \to p$ follows from $(\neg p) \vee q \ \vdash \ p \to q$ simply by a relabelling as seen in question 2.3.