Section 6 Series in General

We now consider series in which some, and in fact possibly infinitely many, of the terms are negative. Given such a series $\sum_{r=1}^{\infty} a_r$ we might first think to examine $\sum_{r=1}^{\infty} |a_r|$, a series of non-negative terms to which we could apply the results of section 5.

The convergence of the two series are related by

Theorem 6.1 Let $\sum_{r=1}^{\infty} a_r$ be a series. If $\sum_{r=1}^{\infty} |a_r|$ is convergent then $\sum_{r=1}^{\infty} a_r$ is convergent.

Proof

Assume $\sum_{r=1}^{\infty} |a_r|$ is convergent.

From the definition of modulus we have that

$$-|a_r| \le a_r \le |a_r| \text{ for all } r \ge 1,$$

and so

$$0 \le a_r + |a_r| \le 2|a_r| \text{ for all } r \ge 1. \tag{12}$$

By Theorem 4.4 we know that we can multiply a convergent series term-by-term by a constant and still have a convergent series, and so in particular $\sum_{r=1}^{\infty} 2|a_r|$ is convergent. Then by the First Comparison Test and (12) we deduce that $\sum_{r=1}^{\infty} (a_r + |a_r|)$ is convergent. Finally we use Theorem 4.4 again, this time it tells us we can add or subtract convergent series to get new convergent series. In particular we can deduce that

$$\sum_{r=1}^{\infty} a_r = \sum_{r=1}^{\infty} ((a_r + |a_r|) - |a_r|)$$

is convergent. ■

Definition A series $\sum_{r=1}^{\infty} a_r$ is called **absolutely convergent** if $\sum_{r=1}^{\infty} |a_r|$ is convergent.

Note We can write Theorem 6.1 as:

$$\text{absolutely convergent} \Rightarrow \text{convergent}.$$

For applications see Question 2, Sheet 6.

In this course we do not come across many series that are not absolutely convergent.

Definition A series is said to be **alternating** if its terms are alternatively positive and negative.

Example

$$\sum_{r=1}^{\infty}(-1)^{r+1}\frac{1}{r} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...$$

is an alternating series.

Theorem 6.2 (Alternating Series Test)

Let $\sum_{r=1}^{\infty}(-1)^{r+1}a_r = a_1 - a_2 + a_3 - a_4 + ...$ be a series with $a_r > 0$ for all r. Suppose that the sequence $\{a_n\}_{n\in\mathbb{N}}$ is decreasing with limit 0.

Then $\sum_{r=1}^{\infty}(-1)^{r+1}a_r$ is convergent.

Proof The proof is not examinable so I have relegated it to the appendix.

Example

Show that $\sum_{r=1}^{\infty}(-1)^{r+1}\frac{1}{r}$ is convergent.

Solution

The sequence $a_r = \frac{1}{r}, r \geq 1$, is obviously decreasing. It is equally obvious that $\lim_{r\to\infty} a_r = 0$.

Hence, by Theorem 6.2, $\sum_{r=1}^{\infty}(-1)^{r+1}\frac{1}{r}$ converges. ■

This example shows that the converse of Theorem 6.1 is FALSE. For we now have an example of a series, namely, $\sum_{r=1}^{\infty}(-1)^{r+1}\frac{1}{r}$ which is convergent but for which

$$\sum_{r=1}^{\infty}\left|(-1)^{r+1}\frac{1}{r}\right| = \sum_{r=1}^{\infty}\frac{1}{r}$$

is divergent, being the Harmonic Series.

Definition A series is said to be **conditionally convergent** if $\sum_{r=1}^{\infty} a_r$ converges yet $\sum_{r=1}^{\infty} |a_r|$ diverges.

So $\sum_{r=1}^{\infty}(-1)^{r+1}\frac{1}{r}$ is a conditionally convergent series.

For more examples see Question 1 Sheet 6

Remember

absolutely convergent $\Rightarrow$ convergent, but convergent $\not\Rightarrow$ absolutely convergent.

Tests for Convergence

Unlike Theorems 5.2 and 5.3 the following convergence tests do **not** require the use of a second series.

Theorem 6.3 (D'Alembert's Ratio Test)

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of non-zero real numbers and suppose the sequence $\left\{\left|\frac{a_{n+1}}{a_n}\right|\right\}_{n\in\mathbb{N}}$ is convergent with limit λ.

(i) If $\lambda < 1$, then $\sum_{r=1}^{\infty} a_r$ converges absolutely.
(ii) If $\lambda > 1$, then $\sum_{r=1}^{\infty} a_r$ diverges.

(If $\lambda = 1$, the test tells us nothing about the series and we need to investigate further.)

Proof

The assumption that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lambda$$

means

$$\forall\varepsilon > 0\ \exists N \in \mathbb{N}\ : \forall n \geq N, \left|\left|\frac{a_{n+1}}{a_n}\right| - \lambda\right| < \varepsilon,$$

i.e.

$$\forall\varepsilon > 0\ \exists N \in \mathbb{N}\ : \forall n \geq N, \lambda - \varepsilon < \left|\frac{a_{n+1}}{a_n}\right| < \lambda + \varepsilon, \tag{13}$$

(i) Assume $\lambda < 1$. Choose $\varepsilon = (1-\lambda)/2$ (which is > 0 since $\lambda < 1$) so that $\lambda + \varepsilon = (1+\lambda)/2 < 1$. Note that $\lambda + \varepsilon > 0$ since $\lambda \geq 0$.

Then from the upper bound in (13) we can find an $N \in \mathbb{N}$ such that

$$|a_{n+1}| < (\lambda + \varepsilon)\,|a_n| \tag{14}$$

for all $n \geq N$. Thus

$$\begin{aligned} |a_{N+1}| &< (\lambda+\varepsilon)|a_N|, \qquad (15)\\ |a_{N+2}| &< (\lambda+\varepsilon)|a_{N+1}| < (\lambda+\varepsilon)^2 |a_N|,\\ |a_{N+3}| &< (\lambda+\varepsilon)|a_{N+2}| < (\lambda+\varepsilon)^3 |a_N|, \end{aligned}$$

In general we prove

$$0 < |a_{N+r}| < (\lambda+\varepsilon)^r |a_N|$$

for all $r \geq 1$ by induction. It holds for $r = 1$ by (15). Assume true for $r = k$. Then consider

$$\begin{aligned} |a_{N+k+1}| &< (\lambda+\varepsilon)|a_{N+k}| && \text{by (14)}\\ &< (\lambda+\varepsilon)(\lambda+\varepsilon)^k |a_N| && \text{by the inductive assumption}\\ &= (\lambda+\varepsilon)^{k+1} |a_N|. \end{aligned}$$

So the result is true for $r = k+1$. Hence result is true for all $r \geq 1$.

Since $|\lambda+\varepsilon| < 1$ the geometric series $\sum_{r=1}^{\infty}(\lambda+\varepsilon)^r |a_N|$ is convergent. Therefore, by the First Comparison Test, Theorem 5.2, $\sum_{r=1}^{\infty}|a_{N+r}| = \sum_{r=N+1}^{\infty}|a_r|$ is convergent and so $\sum_{r=1}^{\infty}|a_r|$ is convergent by Theorem 4.2. Thus $\sum_{r=1}^{\infty} a_r$ is absolutely convergent.

(ii) Assume $\lambda > 1$. Choose $\varepsilon = (\lambda - 1)/2$ (which is > 0 since $\lambda > 1$) so that $\lambda - \varepsilon = (1+\lambda)/2 > 1$.

Then from the lower bound in (13) we can find an $N \in \mathbb{N}$ such that

$$|a_{n+1}| > (\lambda - \varepsilon)|a_n| > |a_n|$$

for all $n \geq N$. Thus

$$|a_N| < |a_{N+1}| < |a_{N+2}| < |a_{N+3}| < \dots .$$

In particular this means that the terms of the series, a_r, do not converge to 0. Hence, by Corollary 4.6, the series diverges. ■

For applications, see Question 9, Sheet 6

Theorem 6.4 (Cauchy's n-th root test.)

Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence and suppose that $\{|a_n|^{1/n}\}_{n\in\mathbb{N}}$ converges with limit λ.

(i) If $\lambda < 1$, then $\sum_{r=1}^{\infty} a_r$ converges absolutely.

(ii) If $\lambda > 1$, then $\sum_{r=1}^{\infty} a_r$ diverges.

(If $\lambda = 1$, this test tells us nothing about the series and we need to investigate further.)

Proof

Let $\varepsilon > 0$ be given. The assumption that $\lim_{n\to\infty} |a_n|^{1/n} = \lambda$ means there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} & \left| |a_n|^{1/n} - \lambda \right| < \varepsilon, \\ \text{i.e} \quad & \lambda - \varepsilon < |a_n|^{1/n} < \lambda + \varepsilon, \\ \text{or} \quad & (\lambda - \varepsilon)^n < |a_n| < (\lambda + \varepsilon)^n , \end{aligned} \tag{16}$$

for all $n > N$.

(i) Assume $\lambda < 1$. Choose $\varepsilon = (1 - \lambda)/2$ (which is > 0 since $\lambda < 1$) so that $\lambda + \varepsilon = (1 + \lambda)/2 < 1$. Note that $\lambda + \varepsilon > 0$ since $\lambda \geq 0$.

By the upper bound in (16) there exists $N_1 \in \mathbb{N}$ such that $|a_n| < (\lambda + \varepsilon)^n$ for all $n \geq N_1$.

Since $\lambda + \varepsilon < 1$ the geometric series $\sum_{r=N_1}^{\infty} (\lambda + \varepsilon)^n$ converges. So, by the First Comparison Test, $\sum_{r=N_1}^{\infty} |a_r|$ converges.

Therefore, $\sum_{r=1}^{\infty} |a_r|$ converges as required.

(ii) Assume $\lambda > 1$. Choose $\varepsilon = (\lambda - 1)/2$ (which is > 0 since $\lambda > 1$) so that $\lambda - \varepsilon = (1 + \lambda)/2 > 1$.

By the lower bound in (16) there exists $N_2 \in \mathbb{N}$ such that $|a_n| > (\lambda + \varepsilon)^n > 1^n = 1$ for all $n \geq N_2$. This means that the sequence $\{|a_n|\}_{n\geq 1}$ does not converge to 0, which in turn means that $\{a_n\}_{n\geq 1}$ does not converge.

Hence, by Corollary 4.5, $\sum_{r=1}^{\infty} a_r$ diverges. ■

For applications see Question 18, Sheet 6

Power Series

Definition Let x be a real number and let $\{a_r\}_{r\geq 0}$ be a sequence. The series

$$\sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...$$

is called a **power series** in x. Note that the series starts at 0 and not 1.

So the geometric series of Section 4 are a particular type of power series, namely $a_i = \lambda$ for all $i \geq 1$.

Let $\mathcal{S} \subseteq \mathbb{R}$ be the set of values of x for which the power series is convergent. (Later we shall see that $\mathcal{S}$ is a special kind of set.)

Example Find those x for which $\sum_{r=0}^{\infty} \frac{x^r}{r!}$ is convergent.

Solution

Let $a_n = \frac{x^n}{n!}$. Then

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}}{(n+1)!}\frac{n!}{|x|^n} = \frac{|x|}{n+1},$$

so

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0 \qquad \text{for all } x.$$

Thus, in the notation of the ratio test $\lambda = 0$ for all x and so $\sum_{r=0}^{\infty} \left|\frac{x^r}{r!}\right|$ converges for all x. Then, by Theorem 6.1, we have that $\sum_{r=0}^{\infty} \frac{x^r}{r!}$ converges for all $x \in \mathbb{R}$. ■

We can define a function exp: $\mathbb{R} \to \mathbb{R}$ by

$$\exp(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!}.$$

By the above example it is seen that this function is well-defined for all $x \in \mathbb{R}$.

For a general power series $\sum_{r=0}^{\infty} a_r x^r$ we can define a function $f : \mathcal{S} \to \mathbb{R}$ by $f(x) = \sum\limits_{r=0}^{\infty} a_r x^r$.

Example Show that

$$\sum_{r=0}^{\infty} (-1)^r \frac{x^{2r+1}}{(2r+1)!} \qquad \text{and} \qquad \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{(2r)!}$$

converge for all $x \in \mathbb{R}$.

Solution

Use exactly the same method as in the above example, namely first use the Ratio Test to show that both

$$\sum_{r=0}^{\infty} \left|\frac{x^{2r+1}}{(2r+1)!}\right| \quad \text{and} \quad \sum_{r=0}^{\infty} \left|\frac{x^{2r}}{(2r)!}\right|$$

converge for all $x \in \mathbb{R}$. Then apply Theorem 6.2 to get the stated result. ■

These series define the functions sin : $\mathbb{R} \to \mathbb{R}$ and cos : $\mathbb{R} \to \mathbb{R}$ respectively.

In the study of $\mathcal{S}$ we will make use of the following fundamental result.

Lemma 6.5 Let $\sum_{r=0}^{\infty} a_r x^r$ be a power series.

(i) If the series converges for $x_0 \in \mathbb{R}$ then it converges absolutely for all x satisfying $-|x_0| < x < |x_0|$.

(ii) If the series diverges for $x_1 \in \mathbb{R}$ then it diverges for all x satisfying either $x < -|x_1|$ or $x > |x_1|$.

Proof (i) If $x_0 = 0$ there is nothing to prove since there are no x satisfying $-0 < x < 0$. So we can assume that $x_0 \neq 0$. By Theorem 4.5, the fact that the series $\sum_{r=0}^{\infty} a_r x_0^r$ converges implies that $\lim_{n\to\infty} a_n x_0^n = 0$.

By Theorem 3.2 this means that $\{a_n x_0^n\}_{n\in\mathbb{N}}$ is bounded, i.e. there exists $M \geq 0$ such that $|a_n x_0^n| \leq M$ for all $n \in \mathbb{N}$.

Let $x : -|x_0| < x < |x_0|$ be given. (The strict inequality is important.)

Let $t = \dfrac{|x|}{|x_0|}$. Then $0 \leq t < 1$ and

$$0 \leq |a_n x^n| = |a_n x_0^n| t^n \leq M t^n$$

for all $n \in \mathbb{N}$. Now apply the First Comparison Test. Since $0 \leq t < 1$ the geometric series $\sum_{r=0}^{\infty} M t^r$ converges and so $\sum_{r=0}^{\infty} |a_r x^r|$ converges, i.e. $\sum_{r=0}^{\infty} a_r x^r$ converges absolutely. This is true for any $x : -|x_0| < x < |x_0|$.

(ii) This is simply (i) rewritten (if (ii) did not hold then we would get a contradiction with (i)). ■

Terminology Let $R > 0$. We call $(-R, R), (-R, R], [-R, R)$ and $[-R, R]$, **intervals about 0 with radius** R. For completeness we call $\{0\}$ the interval with radius 0 and $\mathbb{R} = (-\infty, \infty)$ the interval with infinite radius. The defining features of these intervals are that they are sets such that if x satisfies $|x| < R$ then x is in the set while if $|x| > R$ then x is not in the set.

Theorem 6.6 Let $\sum_{r=0}^{\infty} a_r x^r$ be a power series. Then the set of values of $x \in \mathbb{R}$ for which the series converges (i.e. the set $\mathcal{S}$) is an interval about 0.

Proof

There are three distinct cases:

(i) $\sum_{r=0}^{\infty} a_r x^r$ converges only for $x = 0$,

(ii) $\sum_{r=0}^{\infty} a_r x^r$converges for all $x \in \mathbb{R}$,

(iii) $\sum_{r=0}^{\infty} a_r x^r$ converges for some $x_0 \neq 0$ and diverges for some $x_1 \neq 0$.

If case (i) holds the result follows with radius $R = 0$, and if case (ii) holds then the result follows with R infinite.

Suppose case (iii) holds. Recall that

$$\mathcal{S} = \left\{ x \in \mathbb{R} : \sum_{r=0}^{\infty} a_r x^r \text{ converges} \right\}.$$

By assumption $\mathcal{S} \neq \mathbb{R}$. We also have that $\sum_{r=0}^{\infty} a_r x_1^r$ diverges. Then by Lemma 6.5(ii) $\sum_{r=0}^{\infty} a_r x^r$ diverges for all x with $|x| > |x_1|$. Hence if $x \in \mathcal{S}$ we must have $|x| \leq |x_1|$, i.e. $-|x_1| \leq x \leq |x_1|$. Thus $\mathcal{S}$ is a bounded set, and in particular, bounded above. Trivially $0 \in \mathcal{S}$ and so $\mathcal{S} \neq \emptyset$, i.e. non-empty. Thus, since $\mathbb{R}$ is complete, $\mathcal{S}$ has a least upper bound. Set $R = lub\mathcal{S}$.

We have to show that $\mathcal{S}$ has the properties of being an interval about 0 with radius R, namely that if x satisfies $|x| < R$ then $x \in \mathcal{S}$ while if $|x| > R$ then $x \notin \mathcal{S}$.

Let x satisfy $|x| < R$. This means that $|x|$ is not an upper bound for $\mathcal{S}$ so we can find $z \in \mathcal{S}$ such that $|x| < z \leq R$. (The difference between z and $|x|$ is the lack of modulus!) Since $z \in \mathcal{S}$ we have that $\sum_{r=0}^{\infty} a_r z^r$ converges. But then, since $|x| < z$ implies $-|z| < |x| < |z|$, we can apply Lemma 6.5(i) to deduce that $\sum_{r=0}^{\infty} a_r x^r$ converges (absolutely), i.e. $x \in \mathcal{S}$.

So we have shown: if $|x| < R$ then $x \in \mathcal{S}$.

Let x satisfy $|x| > R$. For a proof by contradiction assume that $x \in \mathcal{S}$, i.e. $\sum_{r=0}^{\infty} a_r x^r$ converges. Choose a real number z such that $|x| > z > R$. Note that $|x| > z > R \geq 0$ implies $-|x| < z < |x|$. So, by Lemma 6.5(i), the facts $x \in \mathcal{S}$ and $-|x| < z\ z < |x|$ mean that$\sum_{r=0}^{\infty} a_r z^r$ converges, i.e. $z \in \mathcal{S}$. Yet this means that $z \leq lub\mathcal{S} = R$, which contradicts $z > R$. So the assumptions if false, hence $x \notin \mathcal{S}$.

So we have shown: if $|x| > R$ then $x \notin \mathcal{S}$.

Hence $\mathcal{S}$ is an interval about 0. ■

Definition The interval of radius R is called the **interval of convergence** and R is called the **radius of convergence**.

Remarks (i) The proof of Theorem 6.6 shows that for x satisfying $|x| < R$ the power series converges *absolutely*.

(ii) To determine R we can often use the Ratio Test or the n^{th}-root test. In the example defining exp above we used the Ratio test as we do again in the following.

Example Determine the radius of convergence of $\sum_{r=1}^{\infty} \frac{x^r}{r}$.

Solution

Let $a_n = \frac{x^n}{n}$. Then

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}}{(n+1)} \frac{n}{|x|^n} = \left(1 - \frac{1}{n+1}\right) |x|,$$

in which case

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = |x| \text{ for all } x.$$

So, in the notation of Theorem 6.3, $\lambda = |x|$. Thus, if $|x| < 1$ when $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$ then by the Ratio Test, the series converges (absolutely).

If $|x| > 1$, in which case $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$, then by the Ratio Test, the series diverges.

Therefore, the radius of convergence is 1. ■

A further application of the ratio test is

Example Determine the radius of convergence of $\sum_{r=0}^{\infty} (3^r + 7^r)x^r$.

Solution

Rough Work

For very large r the coefficient $(3^r + 7^r)$ is essentially the same as 7^r and the series "looks like" $\sum (7x)^r$. This is a geometric series which we know converges if $|7x| < 1$, i.e. $|x| < \frac{1}{7}$. So we might expect the radius of convergence to be $\frac{1}{7}$.

End of rough work.

Let $a_n = (3^n + 7^n)x^n$. Then

$$\frac{|a_{n+1}|}{|a_n|} = \left|\frac{(3^{n+1} + 7^{n+1})x^{n+1}}{(3^n + 7^n)x^n}\right| = \left(\frac{3\left(\frac{3}{7}\right)^n + 7}{\left(\frac{3}{7}\right)^n + 1}\right) |x|$$

so

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 7|x| \quad \text{ for all } x.$$

So, if $|x| < \frac{1}{7}$ then $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$ and thus, by the Ratio Test, the series converges.

If $|x| > \frac{1}{7}$ then $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$ and thus, by the Ratio Test, the series diverges.

Therefore, the radius of convergence is $\frac{1}{7}$. ■

An application of the n^{th}-root test is given in

Example Find the radius of convergence for $\sum_{r=1}^{\infty}\left(3+\frac{7}{r}\right)^r x^r$.

Solution

Here $a_n=\left(3+\frac{7}{n}\right)^n x^n$ so $|a_n|^{1/n}=\left(3+\frac{7}{n}\right)|x|$ and $\lim_{n\to\infty}|a_n|^{1/n}=3\,|x|$. Hence, in the notation of Theorem 6.5, $\lambda=3\,|x|$ and so, from that result, the series converges absolutely if $3|x|<1$, i.e. $|x|<\frac{1}{3}$, and diverges if $3|x|>1$, that is, $|x|>\frac{1}{3}$. Thus the radius of convergence is $\frac{1}{3}$. ■

Note We can use the ratio and n^{th}-root tests to find R but they wont tell us everything about the interval of convergence. In particular they will not tell us what happens at the end points $x=R$ and $x=-R$. You will have to examine these two cases separately.

***Example** Determine the intervals of convergence of $\sum_{r=1}^{\infty}\frac{x^r}{r}$ and $\sum_{r=0}^{\infty}(3^r+7^r)x^r$.

Solution For $\sum_{r=1}^{\infty}\frac{x^r}{r}$ we have to check $x=\pm 1$. When $x=1$ we get the Harmonic Series which diverges. When $x=-1$ we get a series that converges by the alternating series test. Hence the interval of convergence is $[-1,1)$.

For $\sum_{r=0}^{\infty}(3^r+7^r)x^r$ we have to check $x=\pm 1/7$. For both of these values we see that

$$|(3^r+7^r)x^r|=\left|\frac{(3^r+7^r)}{7^r}\right|>1$$

for all $r\geq 1$. In particular the terms of the series do not tend to zero and so the series cannot converge. Hence, the interval of convergence is $\left(-\frac{1}{7},\frac{1}{7}\right)$. ■

Finally we can state a form of the ratio and n^{th}-root tests that are appropriate for power series. To stop confusion with the notation of Theorem 5.7 write our power series as $\sum_{r=0}^{\infty}b_r x^r$.

Theorem 6.7 Assume either, the sequence $\left\{\left|\frac{b_{n+1}}{b_n}\right|\right\}_{n\in\mathbb{N}}$ converges with limit λ or the sequence $\left\{|b_n|^{1/n}\right\}_{n\in\mathbb{N}}$ converges with limit λ.

(i) If $\lambda\neq 0$ and $|x|<\frac{1}{\lambda}$, then $\sum_{r=0}^{\infty}b_r x^r$ converges absolutely.

(ii) If $\lambda\neq 0$ and $|x|>\frac{1}{\lambda}$, then $\sum_{r=0}^{\infty}b_r x^r$ diverges.

(iii) If $\lambda=0$, then $\sum_{r=0}^{\infty}b_r x^r$ converges absolutely for all x.

In particular, if $\lambda \neq 0$ the radius of convergence is $\frac{1}{\lambda}$ while if $\lambda = 0$ the radius is infinite.

Proof (Left to student but see the appendix.)

Appendix

Theorem 6.2 (Alternating Series Test)

Let $\sum_{r=1}^{\infty}(-1)^{r+1}a_r = a_1 - a_2 + a_3 - a_4 + \dots$ be a series with $a_r > 0$ for all r. Suppose that the sequence $\{a_n\}$ is decreasing with limit 0.

Then $\sum_{r=1}^{\infty}(-1)^{r+1}a_r$ is convergent.

Proof Not given in course.

Rough work

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 - a_2 \leq a_1 = s_1, \\ s_3 &= a_1 - a_2 + a_3 \geq a_1 - a_2 = s_2, \\ &\vdots \end{aligned}$$

So the partial sums satisfy $s_1 \geq s_2$, $s_2 \leq s_3$, $s_3 \geq s_4$, $s_4 \leq s_5, \dots$. That is, the sequence of partial sums "jumps" down from s_1, up from s_2, down from s_3, up from s_4, etc. But, since $\lim_{n\to\infty} a_n = 0$ these jumps get smaller and smaller, and the sequence will converge, to a value less than all the s_n with n odd but larger than all the s_n with n even. A method of proof would be a sandwich type argument where we show that

$$\lim_{\substack{n\to\infty \\ n \text{ odd}}} s_n = \lim_{\substack{n\to\infty \\ n \text{ even}}} s_n.$$

End of rough work

Let s_n be the n^{th} partial sum of $\sum_{r=1}^{\infty}(-1)^{r+1}a_r$. Then, for all $m \in \mathbb{N}$ we have for the partial sums of odd length

$$\begin{aligned} s_{2m+1} - s_{2m-1} &= (-1)^{2m+2}a_{2m+1} + (-1)^{2m+1}a_{2m} \\ &= a_{2m+1} - a_{2m} \\ &\leq 0, \qquad \text{since } \{a_n\} \text{ is decreasing.} \end{aligned}$$

Hence $s_{2m+1} \leq s_{2m-1}$ for all $m \in \mathbb{N}$, i.e. $s_1 \geq s_3 \geq s_5 \geq \dots$, a decreasing sequence.

Also, on bracketing,

$$\begin{aligned} s_{2m+1} &= (a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \dots + (a_{2m-1} - a_{2m}) + a_{2m+1} \\ &\geq 0 \qquad \text{since } a_{2n-1} - a_{2n} \geq 0 \text{ for all } n \text{ and } a_{2m+1} \geq 0. \end{aligned}$$

Hence $s_1 \geq s_3 \geq s_5 \geq ...$ is a decreasing sequence bounded below by 0 and so, by Theorem 3.5, it converges to a limit α, say.

Similarly, for the partial sums of even length,

$$\begin{aligned} s_{2m+2} - s_{2m} &= (-1)^{2m+3} a_{2m+2} + (-1)^{2m+2} a_{2m+1} \\ &= -a_{2m+2} + a_{2m+1} \\ &\geq 0, \qquad \text{since } \{a_n\} \text{ is decreasing.} \end{aligned}$$

Also, on bracketing in a different manner to above,

$$\begin{aligned} s_{2m} &= a_1 - (a_2 - a_3) - (a_4 - a_5) - (a_6 - a_7) - ... - (a_{2m-2} - a_{2m-1}) - a_{2m} \\ &\leq a_1, \qquad \text{since } a_{2n-1} - a_{2n} \geq 0 \text{ for all } n \text{ and } a_{2m} \geq 0. \end{aligned}$$

Hence $s_2 \leq s_4 \leq s_6 \leq ...$ is a increasing sequence bounded above by a_1, so by Theorem 3.4 it converges with limit β, say.

We next show that $\alpha = \beta$ by examining

$$\begin{aligned} |\beta - \alpha| &= |\beta - s_{2m} + s_{2m} - s_{2m+1} + s_{2m+1} - \alpha| \\ &\leq |\beta - s_{2m}| + |s_{2m} - s_{2m+1}| + |s_{2m+1} - \alpha| \\ &= |\beta - s_{2m}| + |a_{2m+1}| + |s_{2m+1} - \alpha| . \end{aligned}$$

Let $\varepsilon > 0$ be given.

Then $\lim_{m\to\infty} s_{2m} = \beta$ implies that there exists $N_1 \in \mathbb{N}$ such that

$$|\beta - s_{2m}| < \frac{\varepsilon}{3} \text{ for all } m \geq N_1. \tag{16}$$

Similarly, $\lim_{n\to\infty} a_n = 0$ implies that there exists $N_2 \in \mathbb{N}$ such that

$$|a_n| < \frac{\varepsilon}{3} \text{ for all } n \geq N_2.$$

Finally, $\lim_{m\to\infty} s_{2m+1} = \alpha$ means that there exists $N_3 \in \mathbb{N}$ such that

$$|s_{2m+1} - \alpha| < \frac{\varepsilon}{3} \text{ for all } m \geq N_3. \tag{17}$$

Choose any $m_0 > N = \max(N_1, N_2, N_3)$. Then

$$|\beta - \alpha| \leq |\beta - s_{2m_0}| + |a_{2m_0+1}| + |s_{2m_0+1} - \alpha| < 3\frac{\varepsilon}{3} = \varepsilon.$$

Since this is true for all $\varepsilon > 0$ we must have $|\beta - \alpha| = 0$, that is, $\alpha = \beta$. Call this common value ℓ.

So finally, given $n \geq N$, if n is even we have from (16) that $|s_n - \ell| < \frac{\varepsilon}{3} < \varepsilon$, while if n odd we have from (17) that, $|\ell - s_n| < \frac{\varepsilon}{3} < \varepsilon$. Thus in all cases $|s_n - \ell| < \varepsilon$ and so $\{s_n\}_{n\in\mathbb{N}}$ converges. ■

Theorem 6.7 Assume either, the sequence $\left\{\left|\frac{b_{n+1}}{b_n}\right|\right\}$ converges with limit λ or the sequence $\left\{|b_n|^{1/n}\right\}$ converges with limit λ.

(i) If $\lambda \neq 0$ and $|x| < \frac{1}{\lambda}$, then $\sum_{r=0}^{\infty} b_r x^r$ converges absolutely.

(ii) If $\lambda \neq 0$ and $|x| > \frac{1}{\lambda}$, then $\sum_{r=0}^{\infty} b_r x^r$ diverges.

(iii) If $\lambda = 0$, then $\sum_{r=0}^{\infty} b_r x^r$ converges absolutely for all x.

In particular, if $\lambda \neq 0$ the radius of convergence is $\frac{1}{\lambda}$ while if $\lambda = 0$ the radius is infinite.

Proof (Left to student.)
Assume that $\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lambda$, in which case, $\lim_{n\to\infty}\left|\frac{b_{n+1}x^{n+1}}{b_n x^n}\right| = \lambda\,|x|$. So if either $\lambda = 0$ or $\lambda \neq 0$ and $|x| < \frac{1}{\lambda}$ then $\lim_{n\to\infty}\left|\frac{b_{n+1}x^{n+1}}{b_n x^n}\right| < 1$ and it follows from Theorem 6.3 that the power series converges absolutely. Otherwise, if $\lambda \neq 0$ and $|x| > \frac{1}{\lambda}$ then $\lim_{n\to\infty}\left|\frac{b_{n+1}x^{n+1}}{b_n x^n}\right| < 1$ and by Theorem 6.3 the series diverges.

Alternatively, assume that $\lim_{n\to\infty}|b_n|^{1/n} = \lambda$, in which case, $\lim_{n\to\infty}|b_n x^n|^{1/n} = \lambda\,|x|$. So if either $\lambda = 0$ or $\lambda \neq 0$ and $|x| < \frac{1}{\lambda}$ then $\lim_{n\to\infty}|b_n x^n|^{1/n} < 1$ and it follows from Theorem 6.4 that the power series converges absolutely. Otherwise, if $\lambda \neq 0$ and $|x| > \frac{1}{\lambda}$ then $\lim_{n\to\infty}|b_n|^{1/n} > 1$ and by Theorem 6.4 the series diverges. ■