Section 3 Sequences and Limits, Continued.

Lemma 3.6 Let $\{a_n\}_{n\in\mathbb{N}}$ be a convergent sequence for which $a_n \neq 0$ for all $n \in \mathbb{N}$ and limit $\alpha \neq 0$. Then there exists $N \in \mathbb{N}$ such that

$$\frac{|\alpha|}{2} \leq |a_n| \leq \frac{3\,|\alpha|}{2}$$

for all $n \geq N$.

In particular this result ensures that if the limit is non-zero then, for all sufficiently large n, the a_n are not too close to 0.

Proof Take $\varepsilon = |\alpha|/2 > 0$ in the definition that $\{a_n\}_{n\in\mathbb{N}}$ has limit α. The definition tells us that there exists $N \in \mathbb{N}$ such that

$$|a_n - \alpha| < \frac{|\alpha|}{2}$$

for all $n \geq N$. Thus

$$|a_n| = |\alpha - (\alpha - a_n)| \leq |\alpha| + |\alpha - a_n| \leq \frac{3\,|\alpha|}{2}$$

by the triangle inequality, while

$$|a_n| = |\alpha - (\alpha - a_n)| \geq |\alpha| - |\alpha - a_n| \geq \frac{|\alpha|}{2}$$

by Corollary 1.2. ■

If $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ are sequences we can form a new sequence $\{a_n + b_n\}_{n\in\mathbb{N}}$ whose n-th term is $a_n + b_n$ for all $n \in \mathbb{N}$. Similarly we can form other, new, sequences $\{a_n b_n\}_{n\in\mathbb{N}}$ and $\{2a_n - b_n^2\}_{n\in\mathbb{N}}$ etc.

Theorem 3.7 Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be convergent sequences with $\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} b_n = \beta$. Then

(i) $\{a_n + b_n\}_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty}(a_n + b_n) = \alpha + \beta$,

(ii) $\{a_n b_n\}_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty}(a_n b_n) = \alpha\beta$,

(iii) If $a_n \neq 0$ for all $n \in \mathbb{N}$ and $\alpha \neq 0$ then $\left\{\frac{1}{a_n}\right\}_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty}\left(\frac{1}{a_n}\right) = \frac{1}{\alpha}$.

Proof

Rough work

(i) Here we will examine $|(a_n + b_n) - (\alpha + \beta)| = |(a_n - \alpha) + (b_n - \beta)| \leq |a_n - \alpha| + |b_n - \beta|$ by the triangle inequality. So if we make both terms on the right hand side less than $\varepsilon/2$ then the left hand side will be less that ε as required.

(ii) Here we will examine

$$\begin{aligned} |a_n b_n - \alpha\beta| &= |a_n b_n - \alpha b_n + \alpha b_n - \alpha\beta| \\ &= |(a_n - \alpha) b_n + \alpha(b_n - \beta)| \\ &\le |b_n||a_n - \alpha| + |\alpha||b_n - \beta|. \end{aligned}$$

Again we will make both terms on the RHS less than $\varepsilon/2$. For this we will have to take account of $|b_n|$ (bounded since the sequence is convergent) and $|\alpha|$.

(iii) Here we will examine

$$\left| \frac{1}{a_n} - \frac{1}{\alpha} \right| = \frac{|\alpha - a_n|}{|a_n||\alpha|}.$$

We can make the numerator, $|\alpha - a_n|$, small but we have to make sure that the denominator, in particular $|a_n|$, does not also get too small. This, though, will follow from Lemma 3.6.

End of rough work.

Let $\varepsilon > 0$ be given.

(i) By definition, $\lim_{n\to\infty} a_n = \alpha$ implies that there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - \alpha| < \varepsilon/2 \text{ for all } n \ge N_1.$$

Similarly, $\lim_{n\to\infty} b_n = \beta$ implies that there exists $N_2 \in \mathbb{N}$ such that

$$|b_n - \beta| < \varepsilon/2 \text{ for all } n \ge N_2.$$

Choose $N = \max(N_1, N_2)$. Then for $n \ge N$ we have, from the rough work above,

$$\begin{aligned} |(a_n + b_n) - (\alpha + \beta)| &\le |a_n - \alpha| + |b_n - \beta| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So we have verified the conditions of the definition that allow us to conclude $\lim_{n\to\infty}(a_n + b_n) = \alpha + \beta$.

(ii) Since $\{b_n\}_{n\in\mathbb{N}}$ is convergent it is bounded (Theorem 3.2) so there exists $M > 0$ such that $|b_n| \le M$ for all n. Further, by taking M larger if necessary we can assume that $|M| > |\alpha|$ also. It is important that M is *not* 0.

Then, since $\lim_{n\to\infty} a_n = \alpha$ we have that there exists $N_3 \in \mathbb{N}$ such that

$$|a_n - \alpha| < \frac{\varepsilon}{2M} \qquad \text{for all } n \geq N_3.$$

Also, $\lim_{n\to\infty} b_n = \beta$ implies that there exists $N_4 \in \mathbb{N}$ such that

$$|b_n - \beta| < \frac{\varepsilon}{2M} \qquad \text{for all } n \geq N_4.$$

Choose $N = \max(N_3, N_4)$. Then, for $n \geq N$, we have, from the rough work above,

$$\begin{aligned}
|a_n b_n - \alpha\beta| &\leq |b_n||a_n - \alpha| + |\alpha||b_n - \beta| \\
&\leq M|a_n - \alpha| + |\alpha||b_n - \beta| \\
&< M\frac{\varepsilon}{2M} + |\alpha|\frac{\varepsilon}{2M} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \qquad \text{since } |\alpha| < M, \\
&< \varepsilon,
\end{aligned}$$

Hence $\lim_{n\to\infty}(a_n b_n) = \alpha\beta$.

(iii) Since $\alpha \neq 0$ we can apply Lemma 3.6 to $\{a_n\}_{n\in\mathbb{N}}$ to find N_5 such that

$$\frac{|\alpha|}{2} < |a_n|$$

for all $n \geq N_5$. In which case

$$\left|\frac{1}{a_n} - \frac{1}{\alpha}\right| = \frac{|\alpha - a_n|}{|a_n||\alpha|} < \frac{2|\alpha - a_n|}{|\alpha|^2}$$

holds for all such n. Now, since $\lim_{n\to\infty} a_n = \alpha$ we can find $N_6 \in \mathbb{N}$ such that

$$|a_n - \alpha| < \frac{\varepsilon|\alpha|^2}{2} \qquad \text{for all } n \geq N_6.$$

Choose $N = \max(N_5, N_6)$. Then for all $n \geq N$ we have

$$\left|\frac{1}{a_n} - \frac{1}{\alpha}\right| < \frac{2|\alpha - a_n|}{|\alpha|^2} < \frac{2}{|\alpha|^2} \times \frac{\varepsilon|\alpha|^2}{2} = \varepsilon.$$

Hence $\lim_{n\to\infty}\left(\frac{1}{a_n}\right) = \frac{1}{\alpha}$. ■

Corollary 3.8 Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be convergent sequences with $\lim_{n\to\infty} a_n = \alpha$ and $\lim_{n\to\infty} b_n = \beta$. Then

(i) For any $\lambda \in \mathbb{R}$, $\{\lambda a_n\}_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty}(\lambda a_n) = \lambda\alpha$.

(ii) $\{a_n - b_n\}_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty}(a_n - b_n) = \alpha - \beta$.

(iii) If $b_n \neq 0$ for all $n \in \mathbb{N}$ and $\beta \neq 0$, then $\left\{\frac{a_n}{b_n}\right\}_{n\in\mathbb{N}}$ is convergent with $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{\alpha}{\beta}$.

Proof

(i) $\lambda, \lambda, \lambda, ...$ is a convergent sequence with limit λ. Use Theorem 3.7(ii) with $b_n = \lambda$ for all n.

(ii) By part (i) with $\lambda = -1$, if $\{b_n\}_{n\in\mathbb{N}}$ is convergent then $\{-b_n\}_{n\in\mathbb{N}}$ is convergent. Now use Theorem 3.7(i) with $\{a_n\}_{n\in\mathbb{N}}$ and $\{-b_n\}_{n\in\mathbb{N}}$.

(iii) By Theorem 3.7(iii) $\lim_{n\to\infty}\left(\frac{1}{b_n}\right) = \frac{1}{\beta}$. Now use Theorem 3.7(ii) with $\left\{\frac{1}{b_n}\right\}_{n\in\mathbb{N}}$ and $\{a_n\}_{n\in\mathbb{N}}$. ■

Example Find

$$\lim_{n\to\infty} \frac{4n^3 + 3n}{1 - 5n^3}.$$

Solution Rough work. For very large n the numerator "looks like" $4n^3$ (imagine evaluating $4n^3 + 3n$ with $n = 10^{12}$ on your calculator, to the significant figures shown on the calculator you would get the same result evaluating $4n^3$ with $n = 10^{12}$ on your calculator). Similarly, for very large n the denominator "looks like" $-5n^3$. Hence for very large n the quotient "looks like" $4n^3/(-5n^3) = -4/5$. So we might guess that the limit is $-4/5$.

Next, we might try to use Corollary 3.8(iii) to prove this. Our first attempt might be to say

$$\lim_{n\to\infty} \frac{4n^3 + 3n}{1 - 5n^3} = \frac{\lim\limits_{n\to\infty}(4n^3 + 3n)}{\lim\limits_{n\to\infty}(1 - 5n^3)}.$$

But we would be **wrong**. This is because Corollary 3.8(iii) says that

"**if** the *two sequences in the quotient converge individually*
then the limit of the quotient is the same as the quotient of the limits".

In our case the two sequences are $\{4n^3 + 3n\}_{n\in\mathbb{N}}$ and $\{1 - 5n^3\}_{n\in\mathbb{N}}$, neither of which converge. So, **before** an application of Corollary 3.8 we have to "change" these two sequences into ones that converge. The usual way is to divide the top and bottom of the quotient by the largest power of n in the bottom term (denominator). In the case, this means dividing top and bottom by n^3 to get sequences $\{4+3/n^2\}_{n\in\mathbb{N}}$ and $\{1/n^3-5\}_{n\in\mathbb{N}}$, both of which converge.

End of rough work.

From an earlier example we know that $\lim_{n\to\infty}\frac{1}{n}=0$. We will use this fact as follows.

$$\begin{aligned}
\lim_{n\to\infty}\frac{4n^3+3n}{1-5n^3} &= \lim_{n\to\infty}\frac{4+\frac{3}{n^2}}{\frac{1}{n^3}-5} && \left(\begin{array}{l}\text{The idea is to get terms on top and}\\ \text{bottom that have finite limits,}\end{array}\right)\\
&= \frac{\lim\limits_{n\to\infty}\left(4+\frac{3}{n^2}\right)}{\lim\limits_{n\to\infty}\left(\frac{1}{n^3}-5\right)} && \text{(Corollary 3.8(iii))}\\
&= \frac{\lim\limits_{n\to\infty}4+\lim\limits_{n\to\infty}\left(\frac{3}{n^2}\right)}{\lim\limits_{n\to\infty}\left(\frac{1}{n^3}\right)-\lim\limits_{n\to\infty}5} && \text{(Theorem 3.7 and Corollary 3.8)}\\
&= \frac{4+3\left(\lim\limits_{n\to\infty}\frac{1}{n}\right)^2}{\left(\lim\limits_{n\to\infty}\frac{1}{n}\right)^3-5} && \text{(Corollary 3.8(iii))}\\
&= \frac{4}{-5}=-\frac{4}{5}.
\end{aligned}$$

■

See also Question 8 Sheet 3

Example Let $\{a_n\}_{n\in\mathbb{N}}$ be the sequence defined recursively by $a_1=0$, and $a_{n+1}=\frac{1}{4}(a_n+1)$ for all $n\in\mathbb{N}$.

Show that this sequence is convergent and find its limit.

Solution Rough work. The first few terms are $0,\frac{1}{4},\frac{5}{16},\frac{21}{64},\frac{85}{256},\ldots$. Looking at these we might guess that the sequence is increasing since

$$0<\frac{1}{4}=\frac{4}{16}<\frac{5}{16}=\frac{20}{64}<\frac{21}{64}=\frac{84}{256}<\frac{85}{256}\ldots .$$

We might also guess that the sequence is bounded above by 1, simply because the numerator of each term is smaller than the denominator. Hence we should be thinking of using Theorem 3.4.

End of Rough work

(i) We will show by induction that $a_n \le a_{n+1} \le 1$ holds for all $n \ge 1$.

If $n = 1$ then $a_1 = 0$ and $a_2 = \frac{1}{4}$ in which case $a_1 \le a_2 \le 1$ holds as required.

Assume true for $n = k$, so we are assuming that $a_k \le a_{k+1} \le 1$ holds. (We hope to show that $a_{k+1} \le a_{k+2} \le 1$ holds.)
Then

$$\begin{array}{ll} a_k \le a_{k+1} \le 1, & \text{by assumption,} \\ a_k + 1 \le a_{k+1} + 1 \le 1 + 1 & \text{adding 1 to each term,} \\ \frac{1}{4}(a_k + 1) \le \frac{1}{4}(a_{k+1} + 1) \le \frac{2}{4} & \text{dividing each term by 4,} \\ a_{k+1} \le a_{k+2} \le \frac{1}{2} < 1 & \text{by the inductive definition of the sequence.} \end{array}$$

So, $a_{k+1} \le a_{k+2} \le 1$, i.e. the result is true for $n = k + 1$.

Thus, by induction, $a_n \le a_{n+1} \le 1$ for all $n \ge 1$.

So, $\{a_n\}_{n\in\mathbb{N}}$ is increasing (since $a_n \le a_{n+1}$ for all $n \ge 1$) and bounded above by 1 (since $a_n \le 1$ for all $n \ge 1$). Hence, by Theorem 3.4, $\{a_n\}_{n\in\mathbb{N}}$ converges, with limit α, say.

(ii) To find α we note that the sequences $0, \frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \ldots$ and $\frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \ldots$ have the same limit, i.e. both $\{a_n\}_{n\ge 1}$ and $\{a_{n+1}\}_{n\in\mathbb{N}} = \left\{\frac{1}{4}(a_n + 1)\right\}_{n\in\mathbb{N}}$ have the same limit. Hence

$$\begin{aligned} \alpha &= \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{4}(a_n + 1) \\ &= \frac{1}{4}\left(\lim_{n\to\infty} a_n\right) + \frac{1}{4} \qquad \text{by Theorem 3.7 and Corollary 3.8,} \\ &= \frac{\alpha}{4} + \frac{1}{4}. \end{aligned}$$

From this we see that $\alpha = 1/3$ as given in the theorem. ■

Note In this problem we had to choose a number λ and then check that $a_n \le \lambda$ for all $n \ge 1$. Here we chose $\lambda = 1$. How do we find λ? We should do some rough work first, *assume* that the limit exists and run through the argument in part (ii) above to find $\alpha = 1/3$. We choose any $\lambda \ge 1/3$, usually a simply value such, as in this case, 1.

See also Questions 9 and 10 Sheet 3

In the proof of theorem 3.10 we will make use of the following result.

Lemma 3.9 (Bernoulli's inequality.)

For all $\delta \geq -1$ and all $n \geq 1$ we have $(1+\delta)^n \geq 1 + n\delta$.

Proof by induction on n.

If $n = 1$ then we get equality.

Assume true for $n = k$, so $(1+\delta)^k \geq 1 + k\delta$. (We now try to show that $(1+\delta)^{k+1} \geq 1 + (k+1)\delta$ holds.)

Consider

$$\begin{aligned} (1+\delta)^{k+1} &= (1+\delta)(1+\delta)^k \\ &\geq (1+\delta)(1+k\delta) \qquad \text{by inductive assumption and } 1+\delta \geq 0, \\ &= 1 + \delta + k\delta + k\delta^2 \\ &> 1 + (k+1)\delta, \qquad \text{dropping the } k\delta^2 \geq 0 \text{ term.} \end{aligned}$$

Thus the result holds for $n = k+1$.

Hence, by induction, $(1+\delta)^n \geq 1 + n\delta$ for all $n \geq 1$. ∎

See also Question 8 Sheet 4

Theorem 3.10

(i) For a fixed integer k,

if $k < 0$, then $\{n^k\}_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} n^k = 0$,

if $k = 0$, then $\{n^k\}_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} n^k = 1$,

if $k > 0$, then $\{n^k\}_{n\in\mathbb{N}}$ is divergent.

(ii) For a fixed real number x,

if $|x| < 1$, then $\{x^n\}_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} x^n = 0$,

if $x = 1$, then $\{x^n\}_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} x^n = 1$,

if $x = -1$ or $|x| > 1$, then $\{x^n\}_{n\in\mathbb{N}}$ is divergent.

Proof

(i) (Left to student.)

(ii) Suppose $|x| < 1$. Let $\varepsilon > 0$ be given.

If $x = 0$ then the result is clear. So suppose $0 < |x| < 1$. In this case we use a TRICK and look at $\dfrac{1}{|x|}$ which satisfies $\dfrac{1}{|x|} > 1$. Write $\dfrac{1}{|x|} = 1 + \delta$ with $\delta > 0$.

By the Archimedean property we can find $N \in \mathbb{N}$ with $\frac{1}{N} < \delta\varepsilon$. Then, for all $n \geq N$,

$$\begin{aligned} |x^n - 0| &= |x|^n = \frac{1}{(1+\delta)^n} \le \frac{1}{1+n\delta}, \qquad \text{by Lemma 3.9,} \\ &< \frac{1}{n\delta} \le \frac{1}{N\delta} < \frac{\delta\varepsilon}{\delta} = \varepsilon. \end{aligned}$$

Therefore $\lim_{n\to\infty} x^n = 0$.

Suppose $x = 1$ then $x^n = 1$ for all n and $\lim_{n\to\infty} x^n = 1$ is immediate.

Suppose $x = -1$ then the sequence is $-1, 1, -1, 1, -1, 1, ...$ which alternates and does not converge, i.e. it diverges.

Suppose $|x| > 1$. Write $|x| = 1 + \delta$ with $\delta > 0$. Let $\lambda > 0$ be given. By the Alternative Archimedean property, Theorem 2.1´, we can find $N \in \mathbb{N}$ with $N > \lambda/\delta$. Then

$$\begin{aligned} \left|x^N\right| &= (1+\delta)^N \ge 1 + N\delta \qquad \text{by Lemma 3.9,} \\ &\ge N\delta > \frac{\lambda}{\delta}\delta = \lambda. \end{aligned}$$

Thus λ is not an upper bound for the sequence $\{x^n\}_{n\in\mathbb{N}}$. But λ was arbitrary so there is no upper bound for $\{x^n\}_{n\in\mathbb{N}}$, i.e. the sequence is unbounded. Therefore, by Corollary 3.3, the sequence is divergent. ■

Theorem 3.11 (Sandwich Rule) Let $\{b_n\}_{n\in\mathbb{N}}, \{c_n\}_{n\in\mathbb{N}}$ be convergent sequences with the same limit ℓ and $b_n \le c_n$ for all $n \in \mathbb{N}$. Suppose $\{a_n\}_{n\in\mathbb{N}}$ satisfies

$$b_n \le a_n \le c_n \text{ for all } n \in \mathbb{N}.$$

Then $\{a_n\}_{n\in\mathbb{N}}$ is convergent with limit ℓ.

Proof

Let $\varepsilon > 0$ be given.

Since$\{b_n\}_{n\in\mathbb{N}}$ is convergent with limit ℓ there exists $N_1 \in \mathbb{N}$ such that $|b_n - \ell| \le \varepsilon$ for all $n \ge N_1$. This means $\ell - \varepsilon < b_n < \ell + \varepsilon$. In particular,

$$\ell - \varepsilon < b_n \tag{1}$$

for all $n \ge N_1$.

Since$\{c_n\}_{n\in\mathbb{N}}$ is convergent with limit ℓ there exists $N_2 \in \mathbb{N}$ such that $|c_n - \ell| \le \varepsilon$ for all $n \ge N_2$. In particular,

$$c_n < \ell + \varepsilon \tag{2}$$

for all $n \geq N_2$.

Let $N = \max(N_1, N_2)$. Then for all $n \geq N$ both (1) and (2) hold and so

$$\ell - \varepsilon < b_n \leq a_n \leq c_n < \ell + \varepsilon.$$

Thus $|a_n - \ell| < \varepsilon$ for all $n \geq N$ as required. ■.

Example Show that

$$\lim_{n\to\infty} \frac{\cos\left(\frac{\pi}{4}n^2\right)}{n^2} = 0.$$

Solution For all $n \geq 1$ we have

$$-1 \leq \cos\left(\frac{\pi}{4}n^2\right) \leq 1$$

and so

$$-\frac{1}{n^2} \leq \frac{\cos\left(\frac{\pi}{4}n^2\right)}{n^2} \leq \frac{1}{n^2}.$$

Thus the terms of our sequence are "sandwiched" by the sequences $-\frac{1}{n^2}$ and $\frac{1}{n^2}$ both of which have limit 0. Hence, Theorem 3.11 gives the result. ■

See also Question 1 Sheet 4

Note We might try to use Theorem 3.7(ii) and say

$$\begin{aligned}
\lim_{n\to\infty} \frac{\cos n}{n^2} &= \lim_{n\to\infty}\left(\frac{1}{n^2}\right) \lim_{n\to\infty}\left(\cos\left(\frac{\pi}{4}n^2\right)\right) \\
&= 0 \times \lim_{n\to\infty}\left(\cos\left(\frac{\pi}{4}n^2\right)\right) \\
&= 0.
\end{aligned}$$

But we would be **wrong**! This is because Theorem 3.7(ii) says that

> "**if** the *two sequences in the product converge individually*
> **then** the limit of the product is the product of the limits".

In our case the two sequences are $\{1/n^2\}_{n\in\mathbb{N}}$ and $\left\{\cos\left(\frac{\pi}{4}n^2\right)\right\}_{n\in\mathbb{N}}$, the second of which does **not** converge. Thus we **cannot** apply Theorem 3.7.

Example Show that

$$\lim_{n\to\infty} \frac{n + (-1)^n}{n - (-1)^n} = 1.$$

Solution For all $n \geq 2$ we have

$$\frac{n-1}{n+1} \leq \frac{n+(-1)^n}{n-(-1)^n} \leq \frac{n+1}{n-1}.$$

So the terms of our sequence are "sandwiched" by the sequences $\frac{1-1/n}{1+1/n}$ and $\frac{1+1/n}{1-1/n}$ both of which have limit 1. Hence, Theorem 3.11 gives the result.■

Appendix

Theorem 3.10

(i) For a fixed integer k,

if $k < 0$, then $\{n^k\}$ is convergent and $\lim_{n\to\infty} n^k = 0$,

if $k = 0$, then $\{n^k\}$ is convergent and $\lim_{n\to\infty} n^k = 1$,

if $k > 0$, then $\{n^k\}$ is divergent.

Proof (Not for examination.)

(i) Suppose $k < 0$. Let $\varepsilon > 0$ be given.

By the Archimedean property we can find $N \in \mathbb{N}$ with $\frac{1}{N} < \varepsilon$.

Since $k < 0$ and $k \in \mathbb{Z}$ we must, in fact, have $k \leq -1$ and thus $n^k \leq n^{-1}$ for all $n \geq 1$. Hence, for any $n \geq N$ we have

$$|n^k - 0| = n^k \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Therefore $\lim_{n\to\infty} n^k = 0$.

If $k = 0$ then $n^k = n^0 = 1$ for all $n \geq 1$ and so $\lim_{n\to\infty} n^k = 1$.

Suppose $k > 0$. Let $\lambda > 0$ be given.

By the Alternative Archimedean property, Theorem 2.1´, we can find $N \in \mathbb{N}$ with $N > \lambda$.

Since $k > 0$ and $k \in \mathbb{Z}$ we must, in fact, have $k \geq 1$ in which case $N^k \geq N > \lambda$. Thus λ is not an upper bound for $\{n^k\}$. But λ was arbitrary so there is no upper bound for $\{n^k\}$, i.e. the sequence is unbounded. Therefore, by Corollary 3.3, the sequence is divergent. ■

Sufficiently large.

We say that a property, $p(n)$, that depends on a natural number n, *holds for all sufficiently large* n, if there exists $N \in \mathbb{N}$ such that $p(n)$ holds for all $n \geq N$.

There are "sufficiently large" variants of most of our results. For instance I leave it to the student to give a proof of the following: Let $\{b_n\}_{n\in\mathbb{N}}, \{c_n\}_{n\in\mathbb{N}}$ be convergent sequences with the same limit ℓ. Suppose $\{a_n\}_{n\in\mathbb{N}}$ satisfies $b_n \leq a_n \leq c_n$ for all sufficiently large n then $\{a_n\}_{n\in\mathbb{N}}$ is convergent with limit ℓ.