Section 3 Sequences and Limits

Definition A **sequence** of real numbers is an infinite ordered list $a_1, a_2, a_3, a_4, ...$ where, for each $n \in \mathbb{N}$, a_n is a real number. We call a_n the *n-th term* of the sequence.

Usually (but not always) the sequences that arise in practice have a recognisable pattern and can be described by a formula.

Examples Find a formula for a_n in each of the following cases:

(i) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$, $\quad a_n = \frac{1}{n}$ for all $n \in \mathbb{N}$,

(ii) $1, -1, 1, -1, \dots$, $\quad a_n = (-1)^{n+1}$ for all $n \in \mathbb{N}$,

(iii) $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \dots$, $\quad a_n = \frac{2^n - 1}{2^n}$ for all $n \in \mathbb{N}$,

(iv) $2, 2, 2, 0, 0, 0, 0, 0, 0, \dots$, $\quad a_n = 2$ if $n \leq 3$, $a_n = 0$ if $n \geq 4$,

(v) $1, 1, 2, 3, 5, 8, 13, \dots$, $\quad a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$, along with $a_1 = a_2 = 1$.

See also Question 5 Sheet 2.

Conversely we can define a sequence by a formula.

Example Let

$$a_n = \begin{cases} 2^n & \text{if } n \text{ odd} \\ n & \text{if } n \text{ even} \end{cases} \quad \text{for all } n \in \mathbb{N}.$$

Then we get the sequence $2, 2, 8, 4, 32, 6, \dots$.

Exercise for student: Show this formula can be written as

$$a_n = \left(\frac{1 + (-1)^n}{2} \right) n + \left(\frac{1 + (-1)^{n+1}}{2} \right) 2^n.$$

Note A sequence is different to a set of real numbers - the order of the terms is important in a sequence but irrelevant in a set. For instance, the *sequence* $1, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$ is different from the *sequence* $\frac{1}{3}, 1, \frac{1}{4}, \frac{1}{5}, \dots$, even though the *sets* $\left\{1, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\right\}$ and $\left\{\frac{1}{3}, 1, \frac{1}{4}, \frac{1}{5}, \dots\right\}$ are identical.

We denote a sequence $a_1, a_2, a_3, \dots$ by $\{a_n\}_{n \in \mathbb{N}}$ or $\{a_n\}_{n \geq 1}$ or just $\{a_n\}$ if there is no confusion. For example $\left\{\frac{2^n - 1}{2^n}\right\}$ is sequence (iii) above.

The *set* containing the sequence is written as $\{a_n : n \in \mathbb{N}\}$.

Definition A real number ℓ is said to be a **limit** of a sequence $\{a_n\}_{n\in\mathbb{N}}$ if, and only if,

$$\text{for every } \varepsilon > 0, \text{ there exists } N \in \mathbb{N} \text{ such that } |a_n - \ell| < \varepsilon \text{ for all } n \geq N$$

or, in mathematical notation,

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, |a_n - \ell| < \varepsilon.$$

Note. To check, or verify that this definition holds we have to:

(i) Guess the value of the limit ℓ,

(ii) Assume $\varepsilon > 0$ has been given,

(iii) Find $N \in \mathbb{N}$ such that $|a_n - \ell| < \varepsilon$, i.e. $\ell - \varepsilon < a_n < \ell + \varepsilon$ for all $n \geq N$.

We have to be able to find such an N for each and every $\varepsilon > 0$ and, in general, the N *will depend on* ε. So you will often see N written as a function of ε, i.e. $N(\varepsilon)$.

See Questions 8 and 9 Sheet 2

Definition A sequence which has a limit is said to be **convergent**. A sequence with no limit is called **divergent.**

Example The sequence $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ is convergent with limit 0.

Solution This is simply the Archimedean Principle. We have to verify the definition above with $\ell = 0$.

Let $\varepsilon > 0$ be given. (So we have no choice over ε, it can be *any* such number.)

The Archimedean Principle says that we can find $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. But then, for all $n \geq N$ we have

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Hence we have verified the definition with $\ell = 0$ which must, therefore, be a limit of the sequence $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$. ■

The question remains whether the sequence $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ has other limits. Note how in the definition I talked about ℓ being **a** limit, not **the** limit. The following result answers this in the negative.

Theorem 3.1 If a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ has a limit, then this limit is unique.

Proof by contradiction.

We hope to prove "For all convergent sequences the limit is unique".

The negation of this is "There exists at least one convergent sequence which does not have a unique limit".

This is what we assume.

On the basis of this assumption let$\{a_n\}_{n\in\mathbb{N}}$ denote a sequence with more than one limit, two of which are labelled as ℓ_1 and ℓ_2 with $\ell_1 \neq \ell_2$.

Choose $\varepsilon = \frac{1}{3}|\ell_1 - \ell_2|$ which is greater than zero since $\ell_1 \neq \ell_2$.

Since ℓ_1 is a limit of $\{a_n\}_{n\in\mathbb{N}}$ we can apply the definition of limit with our choice of ε to find $N_1 \in \mathbb{N}$ such that

$$|a_n - \ell_1| < \varepsilon \text{ for all } n \geq N_1.$$

Similarly, as ℓ_2 is a limit of $\{a_n\}_{n\in\mathbb{N}}$ we can apply the definition of limit with our choice of ε to find $N_2 \in \mathbb{N}$ such that

$$|a_n - \ell_2| < \varepsilon \text{ for all } n \geq N_2.$$

(There is no reason to assume that in the two uses of the definition of limit we should find the same $N \in \mathbb{N}$ for the different ℓ_1 and ℓ_2. They may well be different which is why I have labelled them differently as N_1 and N_2.)

Choose any $m_0 > \max(N_1, N_2)$, then $|a_{m_0} - \ell_1| < \varepsilon$ and $|a_{m_0} - \ell_2| < \varepsilon$. This shows that ℓ_1 is "close to" a_{m_0} and ℓ_2 is also "close to" a_{m_0}. Hence we must have that ℓ_1 is "close to" ℓ_2. Using the Triangle inequality, Theorem 1.2, we can remove the a_{m_0} in the following way: (TRICK)

$$\begin{aligned}
|\ell_1 - \ell_2| &= |\ell_1 - a_{m_0} + a_{m_0} - \ell_2| && \text{"adding in zero"} \\
&\leq |\ell_1 - a_{m_0}| + |a_{m_0} - \ell_2|, && \text{triangle inequality,} \\
&< \varepsilon + \varepsilon, && \text{by the choice of } m_0, \\
&= 2\varepsilon = \tfrac{2}{3}|\ell_1 - \ell_2|, && \text{by the definition of } \varepsilon.
\end{aligned}$$

So we find that $|\ell_1 - \ell_2|$, which is not zero, satisfies $|\ell_1 - \ell_2| < \frac{2}{3}|\ell_1 - \ell_2|$, which is a contradiction.

Hence our assumption must be false, that is, there does not exist a sequence with more than one limit. Hence for all convergent sequences the limit is unique. ■

Notation Suppose $\{a_n\}_{n\in\mathbb{N}}$ is convergent. Then by Theorem 3.1 the limit is unique and so we can write it as ℓ, say. Then we write $\lim_{n\to\infty} a_n = \ell$ or $\mathcal{L}_{n\to\infty} a_n = \ell$ or $a_n \to \ell$ as $n \to \infty$.

In particular, the above example shows that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Example What is the limit of $\left\{1 + \left(-\frac{1}{2}\right)^n\right\}_{n\in\mathbb{N}}$?

Solution Rough work

The first few terms are: $\frac{1}{2}, \frac{5}{4}, \frac{7}{8}, \frac{17}{16}, \frac{31}{32}, \dots$.

It appears that the terms are getting closer to 1.

To prove this we have to consider $|a_n - 1| = \left|\left(1 + \left(-\frac{1}{2}\right)^n\right) - 1\right| = \left(\frac{1}{2}\right)^n$.

n	1	2	3	4	5	6	7	8	9	10
$\lvert a_n - 1\rvert$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$\frac{1}{512}$	$\frac{1}{1024}$

Let $\varepsilon > 0$ be given. We have to show that there exists $N \in \mathbb{N}$ such that $|a_n - 1| < \varepsilon$ for all $n \geq N$.

Consider some particular choices of ε.

$$\begin{aligned}
\varepsilon = \tfrac{1}{10}: &\quad \text{for all } n \geq 4 \quad |a_n - 1| = \tfrac{1}{2^n} \leq \tfrac{1}{16} < \varepsilon,\\
\varepsilon = \tfrac{1}{100}: &\quad \text{for all } n \geq 7 \quad |a_n - 1| < \tfrac{1}{128} < \varepsilon,\\
\varepsilon = \tfrac{1}{1000}: &\quad \text{for all } n \geq 10 \quad |a_n - 1| < \tfrac{1}{1024} < \varepsilon.
\end{aligned}$$

Note how these values of N, namely 4, 7, 10, etc., get larger as ε gets smaller.

End of rough work

Completion of solution. By the Archimedean property we can find $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. For any $n \in \mathbb{N}$ we have $2^n > n$ and so, for all $n \geq N$ we have

$$|a_n - 1| = \frac{1}{2^n} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

as required. ■

Examples Discuss the convergence or otherwise of the following sequences.

(i)	$2, 2, 2, \ldots$,	convergent	limit 2,
(ii)	$2\frac{1}{2}, 2\frac{1}{3}, 2\frac{1}{4}, \ldots$,	convergent	limit 2,
(iii)	$3+2, 3-\frac{2}{2}, 3+\frac{2}{3}, 3-\frac{2}{4}, \ldots$,	convergent	limit 3,
(iv)	$1, 2, 1, 2, \ldots$,	divergent,	
(v)	$\frac{1}{2}, 1\frac{1}{2}, \frac{1}{3}, 1\frac{1}{3}, \frac{1}{4}, 1\frac{1}{4}, \ldots$,	divergent,	
(vi)	$2, 4, 6, 8, \ldots$,	divergent,	
(vii)	$-1, -4, -9, -25, \ldots$.	divergent.	

Example Show, by using the Archimedean principle to verify the definition, that sequence (iii) has limit 3.

Solution

Rough work

The n^{th} term can be written as

$$a_n = 3 + \frac{(-1)^{n+1}2}{n}.$$

So, $|a_n - 3| = \frac{2}{n}$. We will want to find $N \in \mathbb{N}$ such that $\frac{2}{n} < \varepsilon$ for all $n \geq N$, i.e. $\frac{1}{n} < \frac{\varepsilon}{2}$ for such n. Again we will do this by the Archimedean Principle.

End of Rough work

Proof

Let $\varepsilon > 0$ be given. By the Archimedean property we can find $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$. Then for all $n \geq N$ we have

$$|a_n - 3| = \frac{2}{n} \leq \frac{2}{N} < \varepsilon$$

as required. ■

Definition A *sequence* $\{a_n\}_{n\in\mathbb{N}}$ is said to be **bounded** if the *set* $\{a_n : n \in \mathbb{N}\}$ $= \{a_1, a_2, a_3, a_4, \ldots\}$ is bounded.

Similarly a sequence is said to be bounded above or bounded below if the set is bounded above or bounded below respectively.

Example $1, 2, 1, 2, 1, 2\ldots$ is a bounded sequence.

Theorem 3.2 If $\{a_n\}_{n\in\mathbb{N}}$ is a convergent sequence, then $\{a_n\}_{n\in\mathbb{N}}$ is a bounded sequence.

Proof

Let ℓ be the limit of $\{a_n\}_{n\in\mathbb{N}}$. In the definition of limit choose $\varepsilon = 1$ to find $N \in \mathbb{N}$ such that $|a_n - \ell| < 1$ for all $n \geq N$. Rewriting, this says that

$$\ell - 1 < a_n < \ell + 1, \text{ for all } n \geq N,$$

or that the set $\{a_N, a_{N+1}, a_{N+2}, ...\}$ is bounded.

Yet the set $\{a_1, a_2, a_3, ...a_{N-1}\}$ is bounded, above by $\max\{a_i : 1 \leq 1 \leq N-1\}$ and from below by $\min\{a_i : 1 \leq 1 \leq N-1\}$. These maximum and minimums can be calculated simply because the set is finite.

If A, B are bounded sets then $A \cup B$ is bounded.

(Exercise, prove this, but see also Question 3, sheet 2)

Hence $\{a_1, a_2, a_3, ...a_{N-1}\} \cup \{a_N, a_{N+1}, a_{N+2}, ...\} = \{a_1, a_2, a_3, ...\}$ is bounded as is, therefore, the original sequence. ■

Corollary 3.3 If $\{a_n\}_{n\in\mathbb{N}}$ is an unbounded sequence, then $\{a_n\}_{n\in\mathbb{N}}$ is divergent.

Proof: This is just a restatement of Theorem 3.2.

The statement of Theorem 3.2 is of the form "If p then q", often written as "$p \Rightarrow q$". This has been discussed in the appendix to part 2. We also saw there that we represent the negation of a proposition p as $\neg p$. In other words, $\neg p$ means that p does *not* hold.

If we had both $p \Rightarrow q$ and $\neg q \Rightarrow p$ we could combine to deduce $\neg q \Rightarrow p \Rightarrow q$, i.e. $\neg q \Rightarrow q$. It would be a strange world if, assuming that q does *not* hold we could then deduce that q *did* hold. For this reason we say that $p \Rightarrow q$ and $\neg q \Rightarrow p$ are inconsistent.

Without proof I state that $p \Rightarrow q$ and $\neg q \Rightarrow \neg p$ are consistent. In fact they are *logically equivalent* in that if one statement is false than so is the other and if one is true then so is the other. See the appendix to part 2 for more details of equivalence. We say that $\neg q \Rightarrow \neg p$ is the *contrapositive* of $p \Rightarrow q$. The statement of Corollary 3.3 is simply the contrapositive of Theorem 3.2. ■

Example The sequence $1\frac{1}{2}, 2\frac{1}{3}, 3\frac{1}{4}, 4\frac{1}{5}, ...$ is not bounded above and thus it is divergent.

Proof by contradiction.

Assume the sequence is bounded above by λ, say. By the alternative Archimedean principle, Theorem 2.1´, we can find $n \in \mathbb{N}$ such that $n > \lambda$.

But then $n + \frac{1}{n+1}$ is an element of the sequence satisfying $n + \frac{1}{n+1} > n > \lambda$, which is a contradiction.

Hence our assumption is false, thus the sequence is not bounded above.■

Definition A sequence $\{b_n\}_{n\in\mathbb{N}}$ is called a **subsequence of** $\{a_n\}_{n\in\mathbb{N}}$ if, and only if, all of the terms of $\{b_n\}_{n\in\mathbb{N}}$ occur amongst the terms of $\{a_n\}_{n\in\mathbb{N}}$ *in the same order.*

Examples

(i) $a_n = \frac{1}{n}$, $b_n = \frac{1}{2n}$, so

$$\{a_n\}_{n\in\mathbb{N}} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\right\}$$

and

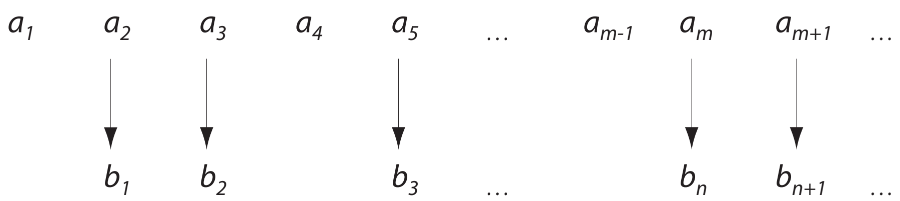
$$\{b_n\}_{n\in\mathbb{N}} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots\right\}$$

which **is** a subsequence of $\{a_n\}_{n\in\mathbb{N}}$.

(ii) $\frac{31}{32}, \frac{63}{64}, \frac{127}{128}, \ldots$, **is** a subsequence of $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \frac{127}{128}, \ldots$.

(iii) $\frac{1}{4}, \frac{1}{2}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots$, is **not** a subsequence of $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots$.

Notes (a) We can look upon a subsequence $\{b_n\}_{n\in\mathbb{N}}$ as the original sequence, $\{a_m\}_{m\in\mathbb{N}}$, with terms deleted and the remaining ones relabelled. For example:



From this we can see that each b_n comes from some a_m where n and m satisfy

$$m = n + (\text{the number of } a_i, 1 \le i \le m-1, \text{that have been deleted}).$$

In particular $m \ge n$. Hence we have

$$\forall n \ge 1, \exists m \ge n : b_n = a_m.$$

The fact that the relabelling retains the ordering means that if $b_n = a_m$ and $b_{n'} = a_{m'}$ then $n \geq n'$ if, and only if, $m \geq m'$.

(b) Example (ii) illustrates the common method of forming a subsequence by omitting a finite number of initial terms of a given sequence.

Theorem 3.4 If a sequence converges then all subsequences converge and all convergent subsequences converge to the same limit.

Proof Let $\{a_n\}_{n\in\mathbb{N}}$ be any convergent sequence. Denote the limit by ℓ.

Let $\{b_n\}_{n\in\mathbb{N}}$ be any subsequence.

Let $\varepsilon > 0$ be given. By the definition of convergence for $\{a_n\}_{n\in\mathbb{N}}$ there exists $N \in \mathbb{N}$ such that $|a_n - \ell| < \varepsilon$ for all $n \geq N$. But this value N will also work for $\{b_n\}_{n\in\mathbb{N}}$. This is because if $n \geq N$ then $b_n = a_m$ for some $m \geq n \geq N$ and so $|b_n - \ell| = |a_m - \ell| < \varepsilon$. Thus $|b_n - \ell| < \varepsilon$ for all $n \geq N$ as required. ■

Question What is the contrapositive of Theorem 3.4?

Question What is the negation of "all subsequences converge and all convergent subsequences converge to the same limit."?

In logic, if it is not the case that both p and q holds then either p does not hold or q does not hold. We could write this as saying "not (p and q)" is logically equivalent to "either (not p) or (not q)". Thus, the negation of "all subsequences converge and all convergent subsequences converge to the same limit" is "either (not all subsequences converge) or (not all convergent subsequences have the same limit)" This is the same as "either (there exists a diverging subsequence) or (there are two converging subsequences with different limits)."

So the contrapositive of Theorem 3.4 is:

Corollary 3.5 If $\{a_n\}_{n\in\mathbb{N}}$ is a sequence that either has a subsequence that diverges or two convergent subsequences with different limits then $\{a_n\}_{n\in\mathbb{N}}$ is divergent.

Example The sequence $1, 2, 1, 2, 1, 2, ...$ is divergent.

Solution Consider the two subsequences $1, 1, 1, ...$ and $2, 2, 2, ...$, both convergent though with different limits, 1 and 2. Hence by the Corollary the sequence $1, 2, 1, 2, 1, 2, ...$ diverges. ■

Example The sequence $1, 2, 3, 1, 2, 3, 1, 2, 3, ...$ is divergent.

Solution Our sequence has a subsequence $1, 2, 1, 2, 1, 2, ...$ which, by the previous example, is divergent. Hence by the Corollary the sequence $1, 2, 3, 1, 2, 3, 1, 2, 3, ...$ diverges. ■

Note The sequence $1, 2, 3, 1, 2, 3, 1, 2, 3, ...$ is bounded but divergent. Thus, $\{a_n\}_{n\in\mathbb{N}}$ being bounded doesn't necessarily mean it is convergent.

Remember these results as

sequence convergent $\Rightarrow$ sequence bounded,
but
sequence bounded $\not\Rightarrow$ sequence convergent.

Aside Something you might try to prove

Theorem Every bounded sequence has a convergent *sub*sequence.

Proof Not given

End of aside.

Definition A sequence $\{a_n\}_{n\in\mathbb{N}}$ is said to be **increasing** (or **non-decreasing**) if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$. (So $a_1 \leq a_2 \leq a_3 \leq a_4 \leq ...$.)

A sequence $\{a_n\}_{n\in\mathbb{N}}$ is said to be **decreasing** (or **non-increasing**) if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. (So $a_1 \geq a_2 \geq a_3 \geq a_4 \geq ...$.)

A **monotone** sequence is one that is either increasing or decreasing.

A sequence is **strictly increasing** if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$, is **strictly decreasing** if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$ and is **strictly monotone** if it is either strictly increasing or strictly decreasing.

Theorem 3.6 Let $\{a_n\}_{n\in\mathbb{N}}$ be a increasing sequence which is bounded above. Then the sequence converges with limit $lub\{a_n : n \in \mathbb{N}\}$.

Proof

The set $\{a_n : n \in \mathbb{N}\}$ is non-empty is bounded above by the assumption of the theorem. So, by the Completeness of $\mathbb{R}$, Property 10, the set has a least upper bound. Denote $lub\{a_n : n \in \mathbb{N}\}$ by β.

We have to verify the definition of convergence with limit β.

Let $\varepsilon > 0$ be given. By Theorem 2.2 there exists $N \in \mathbb{N}$ such that $\beta - \varepsilon < a_N$.

(In words: β is the least of all upper bounds, but $\beta - \varepsilon$ is less than β so cannot be an upper bound and thus must be less than some element in the set.)

Since the sequence is increasing we have

$$\beta - \varepsilon < a_N < a_{N+1} < a_{N+2} < ...,$$

that is, $\beta - \varepsilon < a_n$ for all $n \geq N$.

But β is an upper bound for the set so

$$\beta - \varepsilon < a_n \leq \beta < \beta + \varepsilon \quad \text{or} \quad |a_n - \beta| < \varepsilon$$

for all $n \geq N$.

Thus we have verified the definition of convergence with limit $\beta = lub\{a_n : n \in \mathbb{N}\}$. ■

Theorem 3.7 Let $\{a_n\}_{n\in\mathbb{N}}$ be a decreasing sequence which is bounded below. Then $glb\{a_n : n \in \mathbb{N}\}$ is the limit of $\{a_n\}$ and so, in particular, $\{a_n\}_{n\in\mathbb{N}}$ is convergent.

Proof

Similar to that of Theorem 3.6 and is left as an exercise. ■

Example Let $a_n = \frac{n}{n+1}$ for all n. Show that $\{a_n\}_{n\in\mathbb{N}}$ is convergent.

Solution

Rough work.

Looking at the first few terms $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ...$ they appear to be getting larger. So we might hope to prove

$$\frac{n}{n+1} \leq \frac{n+1}{n+2},$$

i.e. $n(n+2) \leq (n+1)^2$ or $n^2+2n \leq n^2+2n+1$ which is obviously true. We then have to show that the sequence is bounded above and we might guess by 1. So we need $\frac{n}{n+1} \leq 1$, i.e. $n \leq n+1$, again true.

(Again this is not a proof since we have started with what we wanted to prove, deducing true statements, which is the *wrong way round.*)

End of rough work.

Proof

For all $n \in \mathbb{N}$ we have

$$\begin{aligned} & 0 < 1 \\ \Rightarrow\quad & n^2 + 2n \leq n^2 + 2n + 1 \\ \Rightarrow\quad & n(n+2) \leq (n+1)^2 \\ \Rightarrow\quad & \frac{n}{n+1} \leq \frac{n+1}{n+2} \end{aligned}$$

Hence the sequence is increasing.

Also, for all $n \in \mathbb{N}$ we have $n \leq n+1$ in which case $\frac{n}{n+1} \leq 1$. Hence the sequence is bounded above.

Therefore, by Theorem 3.6 the sequence is convergent. ■

Note Using this method we have not found the value of the limit. To do so, we would have to calculate $lub\left\{n/\left(n+1\right):n\in\mathbb{N}\right\}$. The strength of using either Theorem 3.6 or 3.7 is that we do not need to guess the value of the limit.

Appendix

In the appendix to part 2 we discussed "if p then q" or "$p \Rightarrow q$" when p and q are propositions. I said there that the compound proposition $p \Rightarrow q$ is false only when p is True and q False (we never want something false to follow from something true). In all other cases $p \Rightarrow q$ is defined to be True.

Consider now the *contrapositive*, "if not q then not p", or "$(\neg q) \Rightarrow (\neg p)$". When is this False? It is False iff $\neg q$ is True and $\neg p$ False, i.e. iff q is False and p True, i.e. iff $p \Rightarrow q$ is False. So $p \Rightarrow q$ and $(\neg q) \Rightarrow (\neg p)$ are equivalent in that whatever truth values are given to p and q these two compound propositions have the same truth value.

Note, the *converse* of "if p then q" is "if q then p", i.e. the converse of $p \Rightarrow q$ is $q \Rightarrow p$. These are **not** equivalent. For instance, if p is True and q is False then $p \Rightarrow q$ is False while $q \Rightarrow p$ is True.