Section 2 Bounds

Definitions Let $A \subseteq \mathbb{R}$ be a set of real numbers. A real number λ is said to be a **lower bound** (written as lbA) for A if

$$\forall a \in A,\ \ \lambda \leq a.$$

A real number μ is said to be an **upper bound** (written as ubA) for A if

$$\forall a \in A,\ \ a \leq \mu.$$

Note, we do **not** insist that either λ or μ be in A.

If A has a lower bound then we say that A is **bounded below** (denoted by $\underline{bdd}$).

If A has an upper bound then we say that A is **bounded above** (denoted by $\overline{bdd}$).

If A is both bounded above and below then we say that A is **bounded** (denoted by bdd).

If A is not bounded then we say that A is **unbounded.**

Examples Discuss whether the following set are $\underline{bdd}$, $\overline{bdd}$, bdd or unbounded.

(i) $A = \{-2, -1, \frac{1}{2}\}$.

This set is bounded above, by 15 say, and bounded below, by -8 say. So this set is bdd. In fact it is bounded below by every real number $\lambda \leq -2$ and from above by every real number $\mu \geq \frac{1}{2}$.

(ii) $A = (-\infty, \sqrt{2})$.

This set is bounded above by every real number $\mu \geq \sqrt{2}$ and so is $\overline{bdd}$. But the set is not bounded below so it is not $\underline{bdd}$ and so the set is unbounded.

(iii) $A = \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, ...\right\}$.

This set is bounded below by every real number $\lambda \leq \frac{1}{2}$. The set is not bounded above. Hence A is $\underline{bdd}$ and unbounded.

If a set A has a lower bound λ, then every $\lambda^{'} \leq \lambda$ is also a lower bound for A. To see this, let $a \in A$ be given. Since λ is a lower bound for A we have $\lambda \leq a$. We are also told that $\lambda^{'} \leq \lambda$. So, by Property 9 of $\mathbb{R}$, Transitivity, we have $\lambda^{'} \leq a$. True for all $a \in A$ means that $\lambda^{'}$ is a lower bound for A.

Similarly, if a set A has an upper bound μ then every $\mu' \geq \mu$ is also an upper bound for A.

This leads us to consider the *set of all possible lower bounds* for a set A and the *set of all possible upper bounds.*.

Definition Let $A \subseteq \mathbb{R}$. A real number α is said to be the **greatest lower bound** for A (denoted by $glbA$) if, and only if

(i) α is **a** lower bound for A,

(ii) $\alpha \geq \lambda$ for every lower bound λ of A.

(Think about *glb* as the "greatest of all possible lower bounds". Sometimes $glbA$ is written as $\inf A$ for the *infimum* of A.)

Note For the use of the phrase "if, and only if" see the appendix.

Definition Let $A \subseteq \mathbb{R}$. A real number β is called the **least upper bound** ($lubA$) for A, if, and only if,

(i) β is **an** upper bound for A,

(ii) $\beta \leq \mu$ for every upper bound μ of A.

(Think about *lub* as the "least of all possible upper bounds". Sometimes $lubA$ is written as $\sup A$ for the *supremum* of A.)

Examples Find the *glb* and *lub*, if they exist, of the following sets.

(1) $A = \left\{-2, -1, \frac{1}{2}\right\}$.

We might guess that -2 is the greatest lower bound of A. To check that this is so we have to verify the definition with $\alpha = -2$.

(i) Noting that $-2 \leq -2, -2 \leq -1$ and $-2 \leq \frac{1}{2}$ we see that -2 is **a** lower bound for A. So part (i) of the definition is satisfied.

(ii) Given, now, any other lower bound λ of the set, this λ must be less than or equal every element in A. One such element is -2 so we must have $\lambda \leq -2$. If we write this as $-2 \geq \lambda$ we see that part (ii) of the definition is satisfied, i.e. that -2 is greater than or equal to all other lower bounds. Hence $-2 = glbA$.

Note In this example the *glb* **is** an element of the set. This meant the verification of -2 as the *glb* is quite simple.

Similarly $\frac{1}{2}$ is quickly seen to be an upper bound. Given any other upper bound μ this must be greater than or equal the element $\frac{1}{2}$ from the set. In other words $\frac{1}{2} \leq \mu$. Again we have shown that both parts of the definition are satisfied with $\beta = \frac{1}{2}$ and so $\frac{1}{2} = lubA$. ■

(2) $A = (-\infty, \sqrt{2}) = \left\{x \in \mathbb{R} : x < \sqrt{2}\right\}$.

In this case there is no lower bound to the set so no greatest lower bound.

We might guess that $\sqrt{2}$ is the least upper bound for the set. So again we verify the definition of *lub* with $\beta = \sqrt{2}$.

(i) Given any $a \in A$ then by the definition of A we have $a < \sqrt{2}$. Turning this around we have $\sqrt{2} > a$ for all $a \in A$ which simply says that $\sqrt{2}$ is an upper bound for A.

(ii) In this example $\sqrt{2} \notin A$ and so the verification of the definition is not as simple as it was in the last example.

Assume for contradiction that it is **not** true that "for every upper bound μ of A we have.$\sqrt{2} \leq \mu$". This means that there exists at least one upper bound μ_0 say, with $\mu_0 < \sqrt{2}$. Consider $\xi = (\mu_0 + \sqrt{2})/2$, the average of μ_0 and $\sqrt{2}$. Then $\mu_0 < \xi < \sqrt{2}$. There are two bits of information here. Firstly $\xi < \sqrt{2}$ which implies $\xi \in A$ by the definition of A. Secondly $\mu_0 < \xi$. But combining this with $\xi \in A$ we conclude that μ_0 is **not** an upper bound for A. This contradicts the definition of μ_0 as an upper bound for A. So our assumption is false and thus $\sqrt{2} \leq \mu$ for every upper bound μ of A

Thus we have verified both parts (i) and (ii) of the definition of *lub* with $\beta = \sqrt{2}$. ■

(3) $A = \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots \right\}$.

This set has no upper bound and so no least upper bound. As in example (1) you can show that $glbA = \frac{1}{2}$. ■

Note that Example (2) illustrates the point that *lubA* (and *glbA*) do not necessarily belong to A. In general, if the *lub* or *glb* is not in the set, it is harder to verify the definition. The verification is often best done by contradiction, as above.

We can now provide the missing property 10 concerning $\mathbb{R}$:

Property 10 (**Completeness**) Every non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound; and every non-empty subset of $\mathbb{R}$ which is bounded below has a greatest lower bound.

Note: This property should really read: Every non-empty set of *real numbers* which is bounded above *by a real number* has a least upper bound *that is a real number*; and every non-empty set of *real numbers* which is bounded below *by a real number* has a greatest lower bound *that is a real number.*

So if we ask whether this property holds for $\mathbb{Q}$, say, we are asking the following. Is it true that every non-empty set of *rational* numbers which is bounded above *by a rational number* has a least upper bound *that is a rational number*; and every non-empty set of *rational* numbers which is bounded below *by a rational number* has a greatest lower bound *that is a rational number*?

The answer is **no**, and so Property 10 does not hold in $\mathbb{Q}$, unlike all the other Properties $1 - 9$.

Consider $\{x \in \mathbb{Q} : x^2 < 2\}$. This set is bounded above by $2 \in \mathbb{Q}$, for example, but in the following result it is seen that it has no least upper bound in $\mathbb{Q}$ (it does have one in $\mathbb{R}$, as it should by property 10, namely $\sqrt{2}$, but $\sqrt{2} \notin \mathbb{Q}$).

Theorem The least of all rational upper bounds of $\{x \in \mathbb{Q} : x^2 < 2\}$ is not rational.

Proof (This result is not given in the lectures, it will not be examined but is given at the end of this section for the sake of your education.)

The following result is often useful when we wish to prove that a given number is the *lub* or *glb* of a set.

Theorem 2.1 (Archimedean property of $\mathbb{R}$)

$$\forall \varepsilon > 0, \exists n \in \mathbb{N} : \frac{1}{n} < \varepsilon.$$

Proof

Let

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... \right\}.$$

Since $0 < \frac{1}{n}$ for all $n \in \mathbb{N}$ we see that 0 is **a** lower bound for A.

Then, by Property 10, A has a *glb* which we can call α.

Claim: $\alpha = 0$.

Proof of claim. (TRICK)

Since 0 is **a** lower bound for A and α is the **greatest** of all such lower bounds we must have $\alpha \geq 0$.

Now

	$\frac{1}{2n} \in A$	$\forall\ n \in \mathbb{N}$,
so	$\alpha \leq \frac{1}{2n}$	$\forall\ n \in \mathbb{N}$, since α is a lower bound for A,
and thus	$2\alpha \leq \frac{1}{n}$	$\forall\ n \in \mathbb{N}$, on multiplying through by 2.

This means that 2α is **a** lower bound for A. Yet, again, α is the **greatest** of all lower bounds so we must have $\alpha \geq 2\alpha$. Subtracting α from both sides gives $\alpha \leq 0$.

Thus we have both $\alpha \geq 0$ and $\alpha \leq 0$. Hence $\alpha = 0$. ♦

Finally, let $\varepsilon > 0$ be given. Since $0 = glbA$, we have $\varepsilon > glbA$. Thus ε is **not** a lower bound for A and we can find an element of A lower than ε, i.e. there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. ■

Theorem 2.1´ (Alternative Archimedean property)

Given any real number $K > 0$ there exists $n \in \mathbb{N}$ such that $n > K$.

Proof Given any $K > 0$ set $\varepsilon = \frac{1}{K} > 0$ and apply Theorem 2.1 to find $n \in \mathbb{N}$ with $\frac{1}{n} < \varepsilon = \frac{1}{K}$. In this way we find $n \in \mathbb{N}$ with $n > K$. ■

In particular this shows that $\mathbb{N}$ is not bounded above.

Our definitions of *glb* and *lub* can be difficult to apply in practice. Because of this we have alternative forms:

Recall that if the *lub* is not in the set, the verification of the definition could be quite involved. In verifying the following conditions in place of those of the definition, we often avoid the proof by contradiction seen before.

Theorem 2.2 A real number β is the $lubA$ if and only if

(a) β is an upper bound of A

(b) for every $\varepsilon > 0$ there exists $a \in A$ such that $\beta - \varepsilon < a$ $(\leq \beta)$ i.e.

$$\forall \varepsilon > 0, \exists a \in A : \beta - \varepsilon < a.$$

(So (b) says that for every $\varepsilon > 0$, $\beta - \varepsilon$ is *not* an upper bound for A, as we would expect if β is the least of all upper bounds.)

Proof

We need to show that

$$(a)\,,(b) \text{ hold } \Leftrightarrow (i)\,,(ii) \text{ hold.}$$

Yet (a) and (i) are identical. So it suffices to assume that β *is* an upper bound and show that

$$(b) \text{ holds } \Leftrightarrow (ii) \text{ holds.}$$

$(\Leftarrow)$ Assume that the upper bound β satisfies part (ii) of the definition, i.e. $\beta \leq \mu$ for every upper bound μ of A.

Let $\varepsilon > 0$ be given. Trivially $\beta - \varepsilon < \beta$ and so, by the assumption, $\beta - \varepsilon$ cannot be one of the upper bounds, μ, for A. This means that there exists $a \in A$ such that $\beta - \varepsilon < a$, as required for part (b) of the Theorem.

$(\Rightarrow)$ Conversely, assume that the upper bound β satisfies part (b) of the Theorem, i.e. for every $\varepsilon > 0$ there exists $a \in A$ such that $\beta - \varepsilon < a$.

Assume for a contradiction that β does not satisfy part (ii) of the Theorem. So there exists at least one upper bound μ_0 of A for which $\mu_0 < \beta$. Set $\varepsilon = \beta - \mu_0 > 0$. Since β satisfies part (b) of the Theorem there exists $a \in A$ such that $\beta - \varepsilon < a$. But this will give us

$$\begin{aligned}\mu_0 &\geq a \quad \text{since } \mu_0 \text{ an upper bound for } A,\\ &> \beta - \varepsilon\\ &= \beta - (\beta - \mu_0) \text{ by our choice of } \varepsilon,\\ &= \mu_0.\end{aligned}$$

That is, $\mu_0 > \mu_0$. This is a contradiction so our last assumption above is false, i.e. β satisfies part (ii) of the Theorem. ■

Theorem 2.3 A real number α is the $glbA$ if and only if

(a) α is a lower bound of A

(b) for every $\varepsilon > 0$ there exists $a \in A$ such that $(\alpha \leq)\ a < \alpha + \varepsilon$, i.e.

$$\forall \varepsilon > 0, \exists a \in A : a < \alpha + \varepsilon.$$

(This time (b) says that for every $\varepsilon > 0$, $\alpha + \varepsilon$ is *not* a lower bound for A.)

Proof

Similar to that of Theorem 2.2 and left to the student. ■

Example Show that 1 is the least upper bound of

$$A = \left\{ \frac{2^n - 1}{2^n} : n \in \mathbb{N} \right\}.$$

Solution

We will try to verify parts (a) and (b) of Theorem 2.2.

First, Rough work.

$$A = \left\{ \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots \right\},$$

so we might guess that $lubA = 1$. To prove this we will try to use Theorem 2.2. So we need to show that parts (a) and (b) of Theorem 2.2 are satisfied when $\beta = 1$.

So we will need (a) $\frac{2^n-1}{2^n} \leq 1$ for all n, i.e. $2^n - 1 \leq 2^n$.

Also (b), given $\varepsilon > 0$ we need to find n such that $1 - \varepsilon < \frac{2^n-1}{2^n} = 1 - \frac{1}{2^n}$, i.e. $\frac{1}{2^n} < \varepsilon$. But, by the Archimedean Property we can find an n such that $\frac{1}{n} < \varepsilon$ which, along with $n < 2^n$ gives $\frac{1}{2^n} < \varepsilon$.

End of rough work.

Note. This is *not* a proof. We have started with what we want to prove and in both cases (a) and (b) ended with true results. In a proof we *start* with true results and *end* with what we want to prove. But this rough working has, at least, shown us what true results we should start with.

Proof

(a) For all $n \in \mathbb{N}$, we trivially have $2^n - 1 < 2^n$ and so $\frac{2^n-1}{2^n} < 1$ for all such n. Thus 1 is **an** upper bound for A.

(b) Let $\varepsilon > 0$ be given. By the Archimedean Property we can find an $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. As $n < 2^n$ for all $n \geq 1$ (a proof can be given by induction) we find that

$$\begin{aligned} \frac{1}{2^n} &< \frac{1}{n} < \varepsilon \\ \text{or} \quad -\varepsilon &< -\frac{1}{2^n} \\ \text{i.e.} \quad 1-\varepsilon &< 1-\frac{1}{2^n} = \frac{2^n-1}{2^n}. \end{aligned}$$

That is, we have found an element of the set A greater than $1-\varepsilon$.

So both parts (a) and (b) of Theorem 2.2 are satisfied with $\beta = 1$ and, thus, $1 = lubA$. ■

Appendix $\mathbb{Q}$ is not Complete.

Imagine we are in a "Universe" containing only rational numbers. Define $A = \{x \in \mathbb{Q} : x^2 < 2\} \subseteq \mathbb{Q}$. Note how A is defined only using rational numbers, namely 2; we have not defined it as $\left\{x \in \mathbb{Q} : x < \sqrt{2}\right\}$. Possible rational upper bounds for A are 1.5,1.42,1.415,1.4143,1.41422,..., (found by rounding up truncations of the decimal expansion for $\sqrt{2}$). The question is then what can be said of the upper bound of A that is less than or equal to all the rational upper bounds.

Theorem The least of all the rational upper bounds of $\left\{x \in \mathbb{Q} : x^2 < 2\right\}$ is not rational.

Proof (This result is not given in the lectures, it will not be examined and is included here just for interest.)

Assume that the least of all rational upper bounds of $\left\{x \in \mathbb{Q} : x^2 < 2\right\}$ is rational and denote it by α.

Then by Property 6 of the real numbers we have that exactly one of $\alpha^2 < 2, \alpha^2 = 2$ or $\alpha^2 > 2$ holds.

We are assuming that α is rational and we know that $x^2 = 2$ has no rational solution so $\alpha^2 = 2$ is not possible.

Thus we have either $\alpha^2 < 2$ or $\alpha^2 > 2$.

Case 1 $\alpha^2 < 2$.

From Question 7(iv) of Problem Sheet 1 we know that we can find a rational $\beta \in Q$ such that $\alpha^2 < \beta^2 < 2$. Firstly, $\beta^2 < 2$ along with $\beta \in \mathbb{Q}$ means $\beta \in \left\{x \in \mathbb{Q} : x^2 < 2\right\}$. But then $\alpha^2 < \beta^2$, that is $\alpha < \beta$, means that the upper bound α is less than some element of $\left\{x \in \mathbb{Q} : x^2 < 2\right\}$, a contradiction.

Case 2 $\alpha^2 > 2$.

As above we can find a rational $\gamma \in \mathbb{Q}$ such that $\alpha^2 > \gamma^2 > 2$. Firstly, given any $z \in \left\{x \in \mathbb{Q} : x^2 < 2\right\}$ we have $z^2 < 2$ and so $z^2 < 2 < \gamma^2$, or $z < \gamma$. True for all z from the set means that γ is a rational upper bound for $\left\{x \in \mathbb{Q} : x^2 < 2\right\}$. This, along with the fact that $\alpha > \gamma$ contradicts the fact that α is the least of all rational upper bounds.

So in each case we are led to either an impossibility or a contradiction, therefore the original assumption is false. Hence result. ■

You could say that there is something "missing" from $\mathbb{Q}$; when we look for the least of all rational upper bounds for A we cannot find it in $\mathbb{Q}$. Thus it seems reasonable to say that $\mathbb{Q}$ is not complete.

Logic

Use of the phrase "if, and only if": Let p and q be two propositions. Propositions are sentences that propose a fact or facts and to which you can give a truth value of either True or False. Examples of propositions are "e is the fifth letter of the alphabet","Gordon Brown is Prime Minister", "$5 + 7 = 11$". The "Calculus of Propositions" studies the question of assigning truth values to compound propositions when we know the truth values of the basic propositions. Given propositions p and q the immediate compounds that can be made are "p does not hold", "p and q", "p or q" and "if p then q". These are symbolised respectively as "$\neg p$", "$p \wedge q$","$p \vee q$" and "$p \Rightarrow q$". All I'll mention here is that we say that $p \Rightarrow q$ is False only when p is True and q is False (we never want something false to follow from something true). In all other (three) cases $p \Rightarrow q$ is True. In English we have other expressions often used in place of "if p then q", such as "q if p" and " p only if q". We say that the compounds "q if p", "p only if q" and "if p then q" are logically equivalent in that whatever truth values are given to p and q these compound sentences have the same truth values.

We can obviously compound already compounded propositions. The compound $(q \Rightarrow p) \wedge (p \Rightarrow q)$ occurs so frequently it is given a symbol $p \Leftrightarrow q$. Using the equivalences above $p \Leftrightarrow q$ can be written in English as "(p if q) and (p only if q)". This is shortened to "p if, and only if, q". And on the blackboard I shorten it even further to "p iff q". Finally, $p \Leftrightarrow q$ has the truth value True if p and q have the same truth values, i.e. both True or both False, while it is False if p and q have different truth values.

Consider now an example with a Universal set $U = \mathbb{R}$ and sentence $2 + x = 5$. If $x = 3$ we get a true sentence, while for any $x \neq 3$ we get false sentences. So whatever value from the Universe is substituted for the variable x we get a proposition. We say that, with

$U = \mathbb{R}$, $2 + x = 5$ is a predicate. To show its dependence on x we might represent it as $p(x)$.

How to write the definition of *glb* of a set $A \subseteq \mathbb{R}$ in terms of predicates? We first have to find the Universe. The variables will be real numbers α and sets $A \subseteq \mathbb{R}$. So the Universe for the definition will be taken as the set of all ordered pairs, $R \times \mathcal{P}(\mathbb{R})$, where $\mathcal{P}(\mathbb{R})$ is the collection of all subsets of $\mathbb{R}$. For predicates we take $p(\alpha, A)$ to be "the real number α is the $glbA$" while for $q(\alpha, A)$ we take "Parts (i) and (ii) of the definition hold for α and A." Then the definition is written as "$p(\alpha, A)$ if, and only if, $q(\alpha, A)$".

$$\text{For all } \alpha \in \mathbb{R}, A \subseteq \mathbb{R},\ p(\alpha, A) \text{ if, and only if, } q(\alpha, A)$$

As we saw above, this represents

$$\text{``}(\forall\alpha \in \mathbb{R}, A \subseteq \mathbb{R}, p(\alpha, A) \Rightarrow q(\alpha, A)) \wedge (\forall\alpha \in \mathbb{R}, A \subseteq \mathbb{R}, q(\alpha, A) \Rightarrow p(\alpha, A)).$$

So the statement of the definition means two things. If we are told that a real number $\alpha = glbA$ then we know that parts (i) and (ii) hold. This is how the definition is used in the first part of the proof of Theorem 2.2, say. On the other hand, if we are told, or can verify, that α and A satisfy parts (i) and (ii) of the definition then we can say $\alpha = glbA$. This is how the definition is used in the example above that $-2 = glb\{-2, -1, \frac{1}{2}\}$.

Note how not only definitions are written as "if, and only if statements", but so are some theorems, such as 2.2 above. With the same Universe as for the definition of *glb* let $r(\beta, A)$ be the predicate that "β and A satisfy parts (i) and (ii) of Theorem 2.2". Then Theorem 2.2 can be written as

$$\text{``For all } \beta \in \mathbb{R}, A \subseteq \mathbb{R}, p(\beta, A) \text{ if, and only if, } r(\beta, A)\text{''}.$$

This represents

$$\text{``}((\forall\beta \in \mathbb{R}, A \subseteq \mathbb{R}, p(\beta, A) \Rightarrow r(\beta, A)) \wedge ((\forall\beta \in \mathbb{R}, A \subseteq \mathbb{R}, r(\beta, A) \Rightarrow p(\beta, A)).$$

So to prove the Theorem we have to show that both $\forall\beta \in \mathbb{R}, A \subseteq \mathbb{R}, p(\beta, A) \Rightarrow r(\beta, A)$ and $\forall\beta \in \mathbb{R}, A \subseteq \mathbb{R}, r(\beta, A) \Rightarrow p(\beta, A)$ are true, which requires two proofs, just as we saw in the proof of Theorem 2.2.