153 Sequences and Series

Section 0 Introduction

Notation

$\mathbb{N}$ is the set of all natural numbers, $\{1, 2, 3, ...\}$,

$\mathbb{Z}$ is the set of all integers, $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$, so $\mathbb{N}$ is the set of all positive integers,

$\mathbb{N}_0$ is the set of all non-negative integers, $\{0, 1, 2, 3, ...\}$,

$\mathbb{Q}$ is the set of all rationals, $\left\{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\right\}$,

$\mathbb{R}$ is the set of all real numbers,

Note that $\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Sequences

The set of natural numbers can be thought of as those numbers used for counting.

The integers then arise from the operation of subtraction on $\mathbb{N}$.

In turn, the rationals arise from the operation of division on $\mathbb{Z}$.

So the sets $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are well understood. And we can go further when we look upon elements of $\mathbb{Q}$ as solutions of linear equations with coefficients from $\mathbb{Z}$. For instance, $\frac{p}{q} \in \mathbb{Q}$ is the solution of the linear equation $qx - p = 0$. This begs the question of what do we get when we look at equations of higher degree. For example $\sqrt{2}$ is a root of $x^2 - 2 = 0$.

Definition The set of all real roots of all polynomials with **integer** coefficients is known as the set of *algebraic numbers* and is denoted by $\mathbb{A}$.

Note that $\mathbb{Q} \subseteq \mathbb{A} \subseteq \mathbb{R}$. In fact we have seen that $\sqrt{2} \in \mathbb{A}$ while from course 112 we might recall that $\sqrt{2} \notin \mathbb{Q}$. Thus $\mathbb{Q} \subset \mathbb{A}$. It was shown in 1882 that π is **not** the root of any polynomial with rational coefficients so $\pi \notin \mathbb{A}$. Thus $\mathbb{A} \subset \mathbb{R}$. So from $\mathbb{Z}$ we can construct this "large" set $\mathbb{A}$, but we still haven't constructed all of $\mathbb{R}$.

Aside You might try to enlarge $\mathbb{A}$ further by including the real roots of polynomials with coefficients not from $\mathbb{Z}$ but from $\mathbb{A}$. But it transpires that we get no new numbers!

We could look upon a real number as a Decimal Expansion, as in $1 = 1.00000... = 1.\overline{0}$. For some numbers a finite expansion is not sufficient. This is because a number with a finite expansion, i.e. 9.2314, must be rational, in this case $\frac{92314}{10000}$. Thus a non-rational, i.e. irrational, number must have an infinite, or *non-terminating*, expansion. But this begs the question of what we **mean** by an infinite decimal expansion. We know that the irrational $\sqrt{2}$ starts off as 1.4142135... . Truncating this infinite expansion we get

$$\begin{array}{ll} a_1 = 1.4, & \text{an approximation to the first decimal place,} \\ a_2 = 1.41, & \text{an approximation to the second decimal place,} \\ a_3 = 1.414, & \text{an approximation to the third decimal place,} \end{array}$$

etc...

That is, $\sqrt{2}$ is given by an infinite ordered list, known as a *sequence,* of approximations. Being finite expansions each of these approximations is a rational number. These rational approximations seem to be getting "closer and closer" to some fixed value, the *limit* of the sequence. So we can look upon the real number $\sqrt{2}$ as the limit of these rational approximations. Thus a motivation for the study of sequences and limits might be that we could define *real numbers* as the *limits* of *sequences* of *rational numbers.* We will not, in fact, study this particular example in this course but I would hope that it might justify our study of sequences and their limits.

The second part of the course deals with

Series

Consider the infinite series

$$\sum = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots ,$$

where the n^{th} term is $\frac{1}{2^{n-1}}$. In trying to give a numerical value to $\sum$ we presumably would calculate

$$\begin{array}{ll} 1 + \frac{1}{2} & = 1\frac{1}{2} = 1.5, \\ 1 + \frac{1}{2} + \frac{1}{4} & = 1\frac{3}{4} = 1.75 \\ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} & = 1\frac{7}{8} = 1.875, \\ \vdots & \vdots \end{array}$$

These numbers seem to be "getting closer to" 2. So perhaps, in some sense, we can say $\sum = 2$. This needs to be made precise.

Note that in examining the *series* $\sum$ we have looked at the *sequence* 1, 1.5 , $1.75, 1.875, 1.9375, 1.96875, \dots$. It is an important aspect of this course that the results of the first half of the course, on sequences, are applied in the second half to derive results about series.

Now consider the series

$$\sum = 1 + 2 + 3 + 4 + ... \ ,$$

where the n^{th} term is n. This time the associated sequence is $1, 3, 6, 10, ...$ which seems to be getting "arbitrarily large" and, in particular, does not tend to a limit. So perhaps the least we should say is that $\sum$ does not represent a real number.

The fundamental question must be: When does an infinite sum represent a real number?

Section 1. Real Numbers

Having raised the question of what is a real number we now admit that a formal definition of a real number is not given in this course. Rather, the existence of real numbers will be assumed and we will study their properties.

Real numbers can be represented by decimal expansions, e.g. $\pi = 3.14159...$, taking care to remember that some real numbers can have two different decimal expansions, e.g. $1 = 1.00000...$ and $0.\overline{9}$.

In this course we shall **assume** the following 10 properties of $\mathbb{R}$.

Property 1 (**Commutative laws** for addition and multiplication.)

$$\forall a, b \in \mathbb{R}, \ a + b = b + a \text{ and } ab = ba.$$

Property 2 (**Associative laws** for addition and multiplication.)

$$\forall a, b, c \in \mathbb{R}, \ a + (b + c) = (a + b) + c \text{ and } a(bc) = (ab)c.$$

Property 3 (**Distributive law**)

$$\forall a, b, c \in \mathbb{R}, \ a(b + c) = ab + ac.$$

Property 4 (**Additive identity and inverses**)

(i) $\exists 0 \in \mathbb{R} : \forall a \in \mathbb{R}, \ a + 0 = a,$

(ii) $\forall a \in \mathbb{R}, \ \exists b \in \mathbb{R} : a + b = 0.$

We call b the *additive inverse* of a and denote this element as $-a$. For brevity we write $c + (-a)$ as $c - a$. So subtraction is defined as the addition of the additive inverse.

Property 5 (**Multiplicative identity and inverses**.)
$\exists 1 \in \mathbb{R}$:

(i) $1 \neq 0$,
(ii) $\forall a \in \mathbb{R},\ a \times 1 = a$,
(iii) $\forall a \in \mathbb{R}, a \neq 0, \exists b \in \mathbb{R} : ab = 1$.

We call b the *multiplicative inverse* of a and denote this element as a^{-1}. For brevity we write ca^{-1} as $\frac{c}{a}$. So division is defined as multiplication by the multiplicative inverse.

From these properties we can derive further properties.

Derived Property A The additive identity and inverses along with the multiplicative identity and inverses are unique.

Derivation If there are two additive identities 0 and $0'$ then

$$\begin{aligned} 0 &= 0 + 0' && \text{(since } 0' \text{ an identity)} \\ &= 0' && \text{(since } 0 \text{ an identity).} \end{aligned}$$

Hence $0 = 0'$.

Assume that for some $a \in \mathbb{R}$ there are two $b, c \in \mathbb{R}$ such that $a + b = 0$ and $a + c = 0$. Start with

$$\begin{array}{ll} a + b = 0 & \\ (a+b) + c = 0 + c & \text{adding } c \text{ to both sides,} \\ c + (a+b) = c & \text{commuativity and Property 4(i),} \\ (c+a) + b = c & \text{associativity,} \\ (a+c) + b = c & \text{commutativity,} \\ 0 + b = c & \text{assumption,} \\ b = c. & \text{Property 4(i).} \end{array}$$

Hence $b = c$.

I leave it to the student to check the uniqueness of the multiplicative identity and inverse. ■

Remark Properties 1-5 are the properties of a **field** and examples will occur in many other courses. Both $\mathbb{A}$ and $\mathbb{Q}$ are fields, the properties 1-5 holding since they are subsets of $\mathbb{R}$. What has to be proved, though, is that they are both closed under addition and multiplication. That is if $x, y \in \mathbb{Q}$ then $x + y$ and $xy \in \mathbb{Q}$. This is not hard to do. It is harder though to show that if $x, y \in \mathbb{A}$ then $x + y$ and $xy \in \mathbb{A}$.

Fields are interesting because of the interaction between the additive and multiplicative properties.

An aside: recall that a *prime number* is defined as an integer that has no factors other than itself or 1. So primes are defined by a multiplicative property. We know that every integer greater than equal to 2 can be written as a product of primes, unique up to ordering. So we have a multiplicative property of primes.

But what are the additive properties of primes? The Goldbach conjecture, first stated in 1742, asserts that every even integer greater than 2 is the sum of two primes. Though it has been verified up to at least 100,00, this conjecture has evaded all attempts at proof. So the additive properties of multiplicative objects can be hard to fathom. End of the aside.

The number 0 has the *additive* property seen in Property 4, so what is the value of the *multiplication* $0 \times a$ for $a \in \mathbb{R}$?

Derived Property B

$$\forall a \in \mathbb{R},\ 0 \times a = 0.$$

Derivation

Let $a \in \mathbb{R}$ be given. Then

$1 = 1 + 0$	since 0 is the additive identity,
$a \times 1 = a \times (1 + 0)$	multiplying both sides by a,
$a = a \times 1 + a \times 0$	using the distributive law and Property 5,
$a = a + a \times 0.$	since 1 is the multiplicative identity.

Hence $a \times 0$ satisfies the condition of being the additive identity. But by derived property A the additive identity 0 is unique, so we must have $a \times 0 = 0$. Since a was *any* real number we have $0 \times a = 0$ *for all* $a \in \mathbb{R}$. ■

Similarly, for $a \in \mathbb{R}$, $-a$ is defined *additively* by Property 4, as is -1. Can we prove that the *multiplication* $(-1)a$ equals $-a$?

Derived Property C

$$\forall a \in \mathbb{R},\ (-1)a = -a.$$

Derivation

Let $a \in \mathbb{R}$ be given. Then

$1 + (-1) = 0$	since -1 is the additive inverse of 1,
$a(1 + (-1)) = a \times 0$	multiplying both sides by a,
$a + (-1)a = 0$	using the distributive law, $a \times 1 = a$ and derived property B.

So $(-1)a$ satisfies the condition of being the additive inverse of a. But, by derived property A the additive inverse of a is unique, being $-a$. Hence $(-1)a = -a$. ■

In the same way we can show

Derived Property D

$$\forall a, b \in \mathbb{R},\ (-a)b = -(ab).$$

Derivation Left to student. See Question 8 Sheet 1.

If nothing else, these examples show how long it takes to derive 'obvious' results from the properties above.

Note Both $\mathbb{R}$ and $\mathbb{C}$, the set of all complex numbers, are examples of fields. So we need further properties of $\mathbb{R}$ to differentiate it from $\mathbb{C}$. For the remaining properties of $\mathbb{R}$ we assume there is an order relation amongst the elements of $\mathbb{R}$. This we denote by $<$ ("is less than").

Property 6

$$\forall a, b \in \mathbb{R} \textit{exactly one} \text{ of the relations holds} : a = b, a < b \text{ or } b < a.$$

Property 7

$$\forall a, b, c \in \mathbb{R},\ \text{if } a < b \text{ then } a + c < b + c.$$

Property 8

$$\forall a, b \in \mathbb{R} \text{ if } 0 < a \text{ and } 0 < b \text{ then } 0 < ab.$$

Property 9 (**Transitivity**)

$$\forall a, b, c \in \mathbb{R},\ \text{if } a < b \text{ and } b < c \text{ then } a < c.$$

I state without proof that it is *not* possible to construct an order relation on $\mathbb{C}$ that satisfies all these properties. Yet the set of real numbers is not the only set that satisfies Properties 1-9. For instance $\mathbb{Q}$ satisfies these properties. To be able to tell $\mathbb{R}$ and $\mathbb{Q}$ apart we need

Property 10 (**Completeness**) which will be given later.

All other properties of the real numbers can be deduced from these ten. We do not prove this.

Notation. For brevity we write

$$\begin{aligned} a &> b \text{ to mean } b < a, \\ a &\leq b \text{ to mean either } a < b \text{ or } a = b, \\ a &\geq b \text{ to mean either } a > b \text{ or } a = b,\ \text{i.e. either } b < a \text{ or } a = b. \end{aligned}$$

Then we can derive the following further elementary properties: for all $a, b, c, d \in \mathbb{R}$ we have

(i) if $c > 0$ and $a \geq b$ then $ac \geq bc$,

Proof $a \geq b$ implies $a > b$ or $a = b$, that is, $b < a$ or $a = b$. Thus we have two cases.

(1) If $b < a$ then

$b + (-b) < a + (-b)$	adding $-b$ to both sides, and Property 7
$0 < a + (-b)$	by Property 4(ii),
$0 < c(a + (-b))$	Property 8, since $0 < c$,
$0 < ca + c(-b)$	by the distributive law,
$0 < ca + (-(cb))$	by derived Property D,
$0 + cb < (ca + (-(cb))) + cb,$	adding cb to both sides, and Property 7,
$cb < ca + (-(cb)) + cb)$	by Property 4(i) and Associative law,
$cb < ca + 0$	by Property 4(ii),
$cb < ca$	by Property 4(i).

Hence $cb < ca$ which can be written as $ac > bc$.

(2) The second case is $a = b$ which simply means that a and b are two different names for the same real number. So ac and bc are also different names for the same number, hence $ac = bc$.

Combining the two cases we have either $ac > bc$ or $ac = bc$, hence $ac \geq bc$. ■

Similarly we can prove

(ii) if $c \leq 0$ and $a \geq b$ then $ac \leq bc$,

(iii) if $a \geq b$ and $c \geq d$ then $a + c \geq b + d$.

Proofs of (ii) and (iii) are left to student. See Question 9 Sheet 1.

Caution There are traps: from $a \geq b$ and $c \geq d$ we cannot deduce $ac \geq bd$ (for example, take $a = 2$, $b = -1$, $c = 1$ and $d = -3$).

Definition The **modulus** or **absolute value** of a real number, a, is denoted by $|a|$ and is given by the rule

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$$

It is easy to see that

$$|x| \leq c \text{ if, and only if, } -c \leq x \leq c. \tag{1}$$

In particular $|x| \leq c$ contains two pieces of information, namely $x \leq c$ *and* $-c \leq x$.

The definition can be rewritten as $a = |a|$ if $a \geq 0$ and $a = -|a|$ if $a < 0$. In both cases we have

$$-|a| \leq a \leq |a|. \tag{2}$$

Theorem 1.1 (Triangle inequality)

$$\forall a, b \in \mathbb{R} \ |a+b| \leq |a| + |b|.$$

(This says that the absolute value of a sum is less than or equal to the sum of the absolute values. The result also holds if $a, b \in \mathbb{C}$, when $|a|, |b|$ and $|a+b|$ are the lengths of the sides of a triangle, hence the name of the result. When $a, b \in \mathbb{R}$ we still have a triangle but this time it is degenerate.)

Proof

Use (2) twice to get

$$-|a| \leq a \leq |a| \text{ and } -|b| \leq b \leq |b|.$$

Add together to get

$$-(|a| + |b|) \leq a + b \leq |a| + |b|.$$

By (1) this is equivalent to $|a+b| \leq |a| + |b|$. ■

Corollary 1.2 For all $a, b \in \mathbb{R}$ we have $|a| - |b| \leq |a - b|$.

Proof

Start with

$$\begin{aligned} |a| &= |a + (-b + b)|, && \text{a common trick of adding in zero as } -b+b, \\ &= |(a-b) + b| \\ &\leq |a-b| + |b|, && \text{by the triangle inequality.} \end{aligned}$$

Hence $|a| - |b| \leq |a - b|$. ■

Intervals

Notation For $a, b \in \mathbb{R}$ define

$$(a,b) = \{x \in \mathbb{R} : a < x < b\}\,, \qquad [a,b] = \{x \in \mathbb{R} : a \le x \le b\}\,.$$

Here (a,b) is called the **open interval** from a to b, while $[a,b]$ is called the **closed interval** from a to b. (These intervals are empty if $a \ge b$ or $a > b$ respectively.)

Occasionally we use the notation

$$[a,b) = \{x \in \mathbb{R} : a \le x < b\}\,, \qquad (a,b] = \{x \in \mathbb{R} : a < x \le b\}\,;$$

the **closed-open** and **open-closed** intervals respectively, together known as the **half-open intervals**.

We use the symbols $+\infty$ and ∞ in the following notation: for $a \in \mathbb{R}$

$$\begin{aligned} (a,+\infty) &= \{x \in \mathbb{R} : a < x\}\,, & [a,+\infty) &= \{x \in \mathbb{R} : a \le x\}\,, \\ [-\infty,a) &= \{x \in \mathbb{R} : x < a\}\,, & (-\infty,a] &= \{x \in \mathbb{R} : x \le a\}\,, \end{aligned}$$

and sometimes write $(-\infty,\infty) = \mathbb{R}$. These are the **semi-infinite** intervals.

Note $+\infty$ and $-\infty$ are symbols and *not* real numbers! Often write ∞ in place of $+\infty$.

In subsequent chapters we will often come across expressions such as $|x-a| < \varepsilon$, where $a \in \mathbb{R}$ and $\varepsilon > 0$, often thought of as small. But $|x-a| < \varepsilon$ means $-\varepsilon < x - a < \varepsilon$, or $a - \varepsilon < x < a + \varepsilon$ or $x \in (a-\varepsilon, a+\varepsilon)$.

Similarly, $|x - a| \le \varepsilon$ means $a - \varepsilon \le x \le a + \varepsilon$ or $x \in [a-\varepsilon, a+\varepsilon]$.