153 Problem Sheet 4

All questions should be attempted. Those marked with a ** must be handed in for marking by your supervisor. Hopefully the supervisor will have time to cover at least the questions marked with a * or **. Questions marked with a # will be discussed in the problems class. Those marked with H are slightly harder than the others.

1**) Use the Sandwich Rule to show that $\lim_{n\to\infty} a_n = 0$ for the following:

$$\text{(i) } a_n = \frac{\sin^2 n}{n}, \qquad \text{(ii) } a_n = \frac{(-1)^n n}{\sqrt{n^3+1}}.$$

2) Use the Sandwich rule to calculate the limits of the following sequences.

$$\text{(i) } \left\{ \frac{2n + \cos(n\pi/4)}{3n + \sin(n\pi/4)} \right\}_{n\in\mathbb{N}}, \qquad \text{(ii) } \left\{ \frac{1}{n}\sqrt{n^2 + 2(-1)^n n + 1} \right\}_{n\in\mathbb{N}}.$$

3) Calculate the numerical values of the first ten partial sums for each of the following series. That is, calculate s_n for $1 \le n \le 10$ in each of the following.

$$\text{(i) } \sum_{r=1}^{\infty} r\left(\frac{1}{2}\right)^r, \qquad \text{(ii) } \sum_{r=1}^{\infty} \frac{1}{r^2}, \qquad \text{(iii) } \sum_{r=1}^{\infty} \frac{1}{r^3}, \qquad \text{(iv) } \sum_{r=1}^{\infty} \frac{1}{\sqrt{r}}.$$

Can you guess from your calculations whether the series converge or diverge?

4#) For each of the following series, find a formula for the n-th partial sum and state whether the series is convergent or divergent.

$$\text{(i) } \sum_{r=1}^{\infty} (r-2) \qquad \text{(ii) } \sum_{r=1}^{\infty} \left(\frac{2}{3}\right)^r.$$

5#) Use partial fractions, in the manner used in the proof of Theorem 4.3, to verify that the expressions given below for the n-th partial sums of the corresponding series. Deduce that the series converge and find their sums.
(i)

$$\text{For } \sum_{r=1}^{\infty} \frac{1}{(2r-1)(2r+1)} \qquad \text{the partial sum } s_n = \frac{1}{2} - \frac{1}{2(2n+1)},$$

(ii)

For $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r\,(r+1)\,(r+2)}$ the partial sum $s_n=\dfrac{1}{4}-\dfrac{1}{2\,(n+1)\,(n+2)}$,

6#) Let $\{a_r\}_{r\in\mathbb{N}}$ be a convergent *sequence* with limit α. For each $r\in\mathbb{N}$, let $b_r=a_r-a_{r+1}$.

Find an expression for the n-th partial sum of $\sum_{r=1}^{\infty}b_r$.

Prove that this *series* converges with sum $a_1-\alpha$.

How does the result of this question relate to question 5 above?

7) Use (†) from Question 2, Sheet 3, to evaluate the n-th partial sum of

$$\sum_{r=1}^{\infty}\frac{1}{\sqrt{r+1}+\sqrt{r}}.$$

Deduce that the series diverges.

8H#) (i) Following the proof of Lemma 3.9 use induction on n to prove that for all $\delta\geq 0$ and all $n\geq 1$ we have

$$(1+\delta)^n\geq 1+n\delta+\frac{n\,(n-1)}{2}\delta^2.$$

(ii) Start from $(1+\delta)^n\geq n\,(n-1)\,\delta^2/2$ and follow the proof of Theorem 3.10 to prove that, if $|x|<1$, then

$$\lim_{n\to\infty}nx^n=0.$$

9) (i) Sum each of the following:

$$\begin{aligned} x+x^2+x^3+...+x^n\\ x^2+x^3+...+x^n\\ x^3+...+x^n\\ \vdots\; \end{aligned}$$

(ii) Add your results together to find an expression for

$$\sum_{r=1}^{n}rx^r.$$

(iii) For $|x|<1$ prove that $\sum_{r=1}^{\infty}rx^r$ converges and find its sum.

10#) For $k \in \mathbb{N}$ define

$$c_k = \left(1 + \frac{1}{k}\right)^k.$$

(i) Prove that for each $k \in \mathbb{N}$

$$\left(1 + \frac{1}{k+1}\right)\left(1 + \frac{1}{k}\right)^{-1} = 1 - \frac{1}{(k+1)^2}.$$

(ii) Deduce that for each $k \in \mathbb{N}$

$$\frac{c_{k+1}}{c_k} = \left(1 + \frac{1}{k+1}\right)\left(1 - \frac{1}{(k+1)^2}\right)^k.$$

(iii) Apply Bernoulli's Lemma, Lemma 3.9, to get, for each $k \in \mathbb{N}$,

$$\frac{c_{k+1}}{c_k} \geq 1 + \frac{1}{(k+1)^3}.$$

(iv) Deduce that $\{c_k\}_{k \in \mathbb{N}}$ is an increasing sequence.

(v) For any $n \geq 1$ use induction on k to show that

$$\left(1 + \frac{1}{n}\right)^k < 1 + \frac{k}{n} + \frac{k^2}{n^2} \qquad \text{for all } 1 \leq k \leq n.$$

(vi) Deduce that $\{c_k\}_{k \in \mathbb{N}}$ is bounded above.

(vii) Conclude that

$$c = \lim_{k \to \infty} \left(1 + \frac{1}{k}\right)^k$$

exists and $2 < c < 3$.

(The value of the limit is e, the base of the natural logarithm.)