Background Notes Theory of Infinite Products.

**Definition 1** Let \( \{u_n\}_{n \geq 1} \) be a sequence of non-zero numbers, real or complex. Let \( p_n = u_1 u_2 \ldots u_n \) for each \( n \).

If the sequence of partial products \( \{p_n\}_{n \geq 1} \) converges to a non-zero limit \( p \) say, as \( n \to \infty \), we say that the infinite product \( \prod_{r=1}^{\infty} u_r \) converges to \( p \).

If a finite number of the factors \( u_n \) equal 0 and the infinite product obtained by removing these factors converges we say that the infinite product \( \prod_{r=1}^{\infty} u_r \) converges to 0.

Otherwise we say that the product is divergent.

Thus a convergent infinite product is zero if at least one of its factors is zero. We will continue by assuming that if \( \{u_n\}_{n \geq 1} \) contains a finite number of zeros these are removed and the remaining terms relabelled. That is we are assuming \( u_n \neq 0 \) for all \( n \). This means we are only considering infinite products that converge to a non-zero value.

**Problem 2** Show that
\[
\prod_{n=1}^{\infty} \left( 1 + \frac{1}{n} \right) \quad \text{and} \quad \prod_{n=2}^{\infty} \left( 1 - \frac{1}{n} \right)
\]
both diverge.

**Hint** Justify
\[
\prod_{n=1}^{N} \left( 1 + \frac{1}{n} \right) = N + 1 \quad \text{and} \quad \prod_{n=2}^{N} \left( 1 - \frac{1}{n} \right) = \frac{2}{N}.
\]

**Lemma 3** Let \( \{u_n\}_{n \geq 1} \) be a sequence of non-zero real numbers with \( u_n \geq 1 \) for all \( n \geq 1 \). Write \( u_n = 1 + a_n \) so \( a_n \geq 0 \) for all \( n \geq 1 \). If \( \sum_{n=1}^{\infty} a_n \) is convergent then \( \prod_{n=1}^{\infty} (1 + a_n) \) is convergent.

**Proof** From \( 1 + x < \exp x \) for all \( x > 0 \) we get
\[
p_N = \prod_{r=1}^{N} u_r = \prod_{r=1}^{N} (1 + a_r) \leq \prod_{r=1}^{N} \exp a_r = \exp \left( \sum_{r=1}^{N} a_r \right).
\]
Also \( p_n > 0 \) for all \( n \) and
\[
p_n = u_n p_{n-1} \geq p_{n-1}
\]
since \( u_n \geq 1 \) for all \( n \). Thus \( \{p_n\}_{n \geq 1} \) is an increasing sequence, bounded above by \( \exp (\sum_{r=1}^{\infty} a_r) \), and thus converges. This is the definition that the infinite product \( \prod_{n=1}^{\infty} (1 + a_n) \) converges. \( \blacksquare \)
Example 4  The infinite product

\[ \prod_{n=1}^{\infty} \left( 1 + \frac{1}{n^2} \right) \]

converges.

It can be shown that the value of the limit is 3.6760779....

Example 5  Use Lemma 3 with Problem 2 to deduce that the Harmonic Series

\[ \sum_{n=1}^{\infty} \frac{1}{n} \]

diverges.

There are questions that can be asked about Lemma 3.

1. If we don’t demand \( a_n \geq 0 \) is it still true that \( \sum_{n=1}^{\infty} a_n \) converges implies \( \prod_{n=1}^{\infty} (1 + a_n) \) converges? Alternatively, can you find a sequence of real numbers \( \{a_n\}_{n\geq1} \) such that \( \sum_{n=1}^{\infty} a_n \) converges but \( \prod_{n=1}^{\infty} (1 + a_n) \) diverges?

2. Does the converse hold, i.e. is it true that \( \prod_{n=1}^{\infty} (1 + a_n) \) converges implies \( \sum_{n=1}^{\infty} a_n \) converges? Alternatively, can you find a sequence of real numbers \( \{a_n\}_{n\geq1} \) such that \( \prod_{n=1}^{\infty} (1 + a_n) \) converges but \( \sum_{n=1}^{\infty} a_n \) diverges?

Try to solve both of these questions. Then have a look at the Appendix.

These examples highlight the ways in which Lemma 3 cannot be extended. But it can be extended to complex \( a_n \) in the following way.

Theorem 6  GJOJ p. 228 If \( a_n \in \mathbb{C}, a_n \neq -1 \) for all \( n \) and \( \sum_{n=1}^{\infty} |a_n| \) is convergent then \( \prod_{n=1}^{\infty} (1 + a_n) \) is convergent.

Example 7  The infinite product

\[ \prod_{n=2}^{\infty} \left( 1 - \frac{1}{n^2} \right) \]

converges.
It can be shown that the value of this product is 0.5. This rather exact value should make us suspicious and when we look closer we find there is no need to apply Theorem 6, instead note that

$$\prod_{n=2}^{N} \left(1 - \frac{1}{n^2}\right) = \frac{N + 1}{2N},$$

for all $N \geq 2$.

A more important application of Theorem 6 is that

**Example 8**

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

converges for all $z \in \mathbb{C}$, $z \neq \pm 1, \pm 2, ...$.

In fact, the product has the value 0 at $z = \pm 1, \pm 2, ...$ and so is well-defined on all of $\mathbb{C}$. This is important since it can be shown that

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

on $\mathbb{C}$, a well-known result due to Euler.

Another important application of Theorem 6 is in the special case when $a_n = 0$ when $n$ is not a prime. We then have an Euler Product $\prod_p (1 + a_p)$.

**Example 9**

$$\prod_{p} \left(1 - \frac{1}{p^s}\right)$$

converges for $\text{Re} \ s > 1$.

**Theorem 10** A necessary and sufficient condition for the product to converge is that the series $\sum_{n=1}^{\infty} \log (1 + a_n)$ converges.


**Definition 11** We say that the infinite product converges absolutely if the series $\sum_{n=1}^{\infty} \left| \log (1 + a_n) \right|$ converges.
An absolutely convergent product is convergent. Also, the value of an absolutely convergent product is not affected by changing the order in which the factors occur.

**Theorem 12** \( \sum_{n=1}^{\infty} |\log (1 + a_n)| \) converges iff \( \sum_{n=1}^{\infty} |a_n| \) converges.

**Proof** See p.177 of *Lectures on the Theory of Functions of a Complex Variable* referenced above.

**Note**

\[
\prod_{n=1}^{\infty} (1 + |a_n|) = 1 + \sum_{n=1}^{\infty} |a_n| + \ldots \geq \sum_{n=1}^{\infty} |a_n|.
\]

So \( \prod_{n=1}^{\infty} (1 + |a_n|) \) converges implies \( \sum_{n=1}^{\infty} |a_n| \) converges which implies, by Theorem 12, \( \prod_{n=1}^{\infty} (1 + a_n) \) converges absolutely.

In the course we will need to take the logarithm of the Riemann Zeta function \( \zeta (s) \), written as the Euler product

\[
\prod_{p} \left( 1 - \frac{1}{p^s} \right)^{-1}.
\]

Yet in Complex Analysis the logarithm of a product is not necessarily the sum of the logarithms of the factors. Nonetheless, it is possible to take the logarithm of each factor here, i.e. find \( w_p \) such that

\[
e^{w_p} = \left( 1 - \frac{1}{p^s} \right)^{-1}.
\]

This logarithm is only unique up to argument modulo \( 2\pi \). But because the Euler product is absolutely convergent it can be shown that the \( w_p \) can be chosen for each \( p \) such that \( \sum_p w_p \) converges. If \( w \) is the value of this series then \( w \) is a logarithm of the zeta function in that \( e^w = \zeta (s) \).

See GJOJ p.70 who shows that

\[
w_p = \sum_{m=1}^{\infty} \frac{1}{mp^ms}
\]

is the correct choice of the logarithm of each Euler factor in Re \( s > 1 \).
Appendix

**Problem 13** Find a sequence of real numbers \( \{a_n\}_{n \geq 1} \) such that \( \sum_{n=1}^{\infty} a_n \) converges but \( \prod_{n=1}^{\infty} (1 + a_n) \) diverges.

**Solution** If all \( a_n \geq 0 \) then Lemma 3 states that \( \sum_{n=1}^{\infty} a_n \) convergent implies \( \prod_{n=1}^{\infty} (1 + a_n) \) convergent. So we must look for an example amongst sequences \( \{a_n\}_{n \geq 1} \) with some \( a_n < 0 \).

I suggest \( a_n = (-1)^n b_n \) with \( b_n > 0 \) for all \( n \). Recall the Alternating Series Test that says that if \( b_n \to 0 \), as we now assume, then \( \sum_{n=1}^{\infty} (-1)^n b_n \) converges.

Further assume that \( \{b_n\}_{n \geq 1} \) is a sequence decreasing to 0.

Noting the non-standard definition, with the product starting at 3 and not 1, consider

\[
p_{2N} = \prod_{n=3}^{2N} (1 + a_n) = \prod_{n=3}^{2N} (1 + (-1)^n b_n)
\]

\[
= \prod_{r=2}^{N} (1 - b_{2r-1}) (1 + b_{2r}),
\]

having collected together consecutive terms. Next recall that \( \{b_n\}_{n \geq 1} \) is a decreasing sequence so \( b_{2r} \leq b_{2r-1} \) and thus

\[
p_{2N} \leq \prod_{r=2}^{N} (1 - b_{2r-1}) (1 + b_{2r-1}) = \prod_{r=2}^{N} (1 - b_{2r-1}^2),
\]

Recalling the earlier example of a divergent product choose \( b_{2r-1} = 1/\sqrt{r} \). Since we need to define \( b_n \) for all \( n \), not just odd \( n \), choose

\[
b_n = \sqrt{\frac{1}{(n+1)/2}} = \sqrt{\frac{2}{n+1}}, \quad \text{that is,} \quad a_n = (-1)^n \sqrt{\frac{2}{n+1}}.
\]

With this choice (and now we see we require \( r \geq 2 \), equivalent to \( n \geq 3 \) above)

\[
p_{2N} \leq \prod_{r=2}^{N} \left(1 - \frac{1}{r}\right) = \prod_{r=2}^{N} \frac{r-1}{r} = \frac{1}{N}.
\]
For odd subscripts note that $1 + a_{2N+1} = 1 - b_{2N+1} < 1$ so
\[ p_{2N+1} = (1 + a_{2N+1}) p_{2N} \leq p_{2N} \leq \frac{1}{N}. \]

Hence $p_n \to 0$ as $n \to \infty$, the definition that the infinite product $\prod_{n=3}^{\infty} (1 + a_n)$ diverges. \[\blacksquare\]

**Problem 14** Find a sequence $\{a_n\}_{n \geq 1}$ such that $\prod_{n=1}^{\infty} (1 + a_n)$ converges yet $\sum_{n=1}^{\infty} a_n$ diverges.

**Solution** The idea is to use the trick in the previous solution and collect consecutive terms together in the product, when
\[ \prod_{n=1}^{\infty} (1 + a_n) = \prod_{r=1}^{\infty} (1 + a_{2r-1}) (1 + a_{2r}) = \prod_{r=1}^{\infty} (1 + b_r). \]

If $b_r > 0$ for all $r \geq 1$ and $\sum_{r=1}^{\infty} b_r < +\infty$ then, by Lemma 3, the infinite product will converge as required. So we need to find a sequence $\{a_n\}$ for which
\[ \sum_{n=1}^{\infty} a_n = \sum_{r=1}^{\infty} (a_{2r-1} + a_{2r}) \quad (1) \]
is divergent yet $b_r = a_{2r-1} + a_{2r} + a_{2r-1}a_{2r} > 0$ for all $r \geq 1$ and
\[ \sum_{r=1}^{\infty} b_r = \sum_{r=1}^{\infty} (a_{2r-1} + a_{2r} + a_{2r-1}a_{2r}) \quad (2) \]
is convergent.

When you think of divergent series you should think of the Harmonic series. So for (1) think of
\[ a_{2r-1} + a_{2r} = \frac{1}{r} + e_r, \quad (3) \]
for some ‘small’ $e_r$. For the sum in (2) to converge we would need the product $a_{2r-1}a_{2r}$ ‘cancel out’ the $1/r$ of $a_{2r-1} + a_{2r}$, i.e.
\[ a_{2r-1}a_{2r} = \frac{1}{r} + f_r \quad (4) \]
for some \( f_r \) for which \( \sum_{r=1}^{\infty} (e_r + f_r) \) converges. The requirement (4) suggests

\[
a_{2r-1} = -\frac{1}{\sqrt{r}} + g_r \quad \text{and} \quad a_{2r} = \frac{1}{\sqrt{r}} + h_r.
\]

for some ‘small’ \( g_r \) and \( h_r \). To satisfy (3) then requires

\[
g_r + h_r = \frac{1}{r} + e_r. \tag{5}
\]

Combining,

\[
a_{2r-1} + a_{2r} + a_{2r-1}a_{2r} = \left( -\frac{1}{\sqrt{r}} + g_r \right) + \left( \frac{1}{\sqrt{r}} + h_r \right) + \left( -\frac{1}{\sqrt{r}} + g_r \right) \left( \frac{1}{\sqrt{r}} + h_r \right)
\]

\[
= g_r + h_r - \frac{1}{r} + \frac{g_r}{\sqrt{r}} - \frac{h_r}{\sqrt{r}} + g_r h_r
\]

\[
= \frac{1}{r} + e_r - \frac{1}{r} + \frac{g_r}{\sqrt{r}} - \frac{h_r}{\sqrt{r}} + g_r h_r \quad \text{by (5)}
\]

\[
= e_r + g_r - \frac{h_r}{\sqrt{r}} + g_r h_r. \tag{6}
\]

Immediately think of setting \( g_r = h_r \) so the centre terms cancel. To satisfy (5) choose \( g_r = h_r = 1/2r \) in which case \( e_r = 0 \). Then, from (6),

\[
a_{2r-1} + a_{2r} + a_{2r-1}a_{2r} = \frac{1}{4r^2},
\]

which is positive and

\[
\sum_{r=1}^{\infty} (a_{2r-1} + a_{2r} + a_{2r-1}a_{2r}) = \frac{1}{4} \sum_{r=1}^{\infty} \frac{1}{r^2},
\]

which converges as required. Hence choosing

\[
a_n = \begin{cases} 
-\frac{1}{\sqrt{r}} + \frac{1}{2r} & \text{if } n = 2r - 1 \\
\frac{1}{\sqrt{r}} + \frac{1}{2r} & \text{if } n = 2r,
\end{cases}
\]

we have an example where \( \prod_{n=1}^{\infty} (1 + a_n) \) converges yet \( \sum_{n=1}^{\infty} a_n \) diverges. ■