MATH20132 Calculus of Several Variables. 2019

Problems 7 Tangent Spaces & Planes

Planes in \( \mathbb{R}^n \).

1. i. A plane in \( \mathbb{R}^3 \) is given parametrically by

\[
\left\{ \begin{pmatrix} 2x + 4y - 5 \\ 2x + y - 2 \\ 2x - 3y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\}.
\]

Express this plane as

a. a graph

\[
\left\{ \begin{pmatrix} x \\ \phi(x) \end{pmatrix} : x \in \mathbb{R}^2 \right\},
\]

of some function \( \phi : \mathbb{R}^2 \to \mathbb{R} \),

b. a level set,

\[
f^{-1}(0) = \{ x \in \mathbb{R}^3 : f(x) = 0 \}.
\]

for some \( f : \mathbb{R}^3 \to \mathbb{R} \).

ii. Repeat for the parametric set

\[
\left\{ \begin{pmatrix} 2x + 2y - 2 \\ x + y - 1 \\ 2x - 3y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\}.
\]

iii. Repeat for

\[
\left\{ \begin{pmatrix} 4x - 4y + 8 \\ -2x + y - 1 \\ 3x - 4y + 6 \\ 4y - 4 \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \right\},
\]

this time expressing this as a graph of some function \( \phi : \mathbb{R}^2 \to \mathbb{R}^2 \), and then as a level set.

2. i. Let \( M \) be an \( n \times r \) real valued matrix. Prove that \( \{ Mt : t \in \mathbb{R}^r \} \), the image of the linear map \( \mathbf{x} \mapsto M\mathbf{x} \), is a vector subspace of \( \mathbb{R}^n \) spanned by the columns of \( M \).
ii. Let $N$ be an $m \times n$ real valued matrix. Prove that \( \{ x \in \mathbb{R}^n : Nx = 0 \} \), the kernel of the linear map $x \mapsto Nx$, is a vector subspace of $\mathbb{R}^n$.

These two results are used in the proof (not given in lectures) concerning planes. This is relegated to an ‘additional question’ at the end of this sheet.

**Best Affine Approximations.**

Recall that the Best Affine Approximation to a function $f$ at a point $a$ is given by

\[
f(a) + df_a(x - a) = f(a) + Jf(a)(x - a).
\]

3. Write down the Best Affine Approximation to

i. $f(x) = x(x + y)$ at $a = (2, -1)^T$, and what value does the approximation give at $a' = (2.1, -0.9)^T$?

ii. $f(x) = xy + yz + xz$ at $a = (-1, -1, 4)^T$, and what value does the approximation give at $a' = (-0.9, -1.1, 4.1)^T$?

iii. $f(x) = \begin{pmatrix} xy^2 \\ x^2y \end{pmatrix}$ at $a = (2, -3)^T$, and what value does the approximation give at $a' = (1.9, -3.1)^T$?

**Hint** Look back at Sheet 4 for appropriate results.

4. Define the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x) = (x^3 - 2xy^2, x + y)^T$. Show that $f$ locally invertible at $a = (1, -1)^T$.

What is the Best Affine Approximation to the inverse function near $b = f(a) = (-1, 0)^T$?

What approximation does this give to $f^{-1}\left((-0.9, 0.1)^T\right)$?

**Tangent Spaces for Graphs**

5. For each of the following scalar-valued functions $\varphi$, find both a basis for the Tangent Space and the equation of the Tangent Plane to the graph of $\varphi$ for the given point on the graph. For the latter give your answer in the form “the Tangent plane to $\varphi$ at $q$ is the graph of the function $g(x) = ...$.”
i. \( \varphi(x) = 4x^2 + y^2, \quad q = (1, -1)^T \in \mathbb{R}^2, \)

ii. \( \varphi(x) = \sqrt{9 - x^2 - y^2}, \quad q = (2, 1)^T \in \mathbb{R}^2, \)

iii. \( \varphi(x) = 9 - x^2 - y^2, \quad p = (2, -2, 1)^T \in G_\varphi, \)

iv. \( \varphi(x) = 5/(1 + x^2 + 3y^2), \quad p = (1, -1, 1)^T \in G_\varphi. \)

6. Repeat Question 5 for the vector-valued function

\[
\phi(x) = \begin{pmatrix} xy \\ x^2 + y^2 \end{pmatrix}
\]

with \( x = (x, y)^T \in \mathbb{R}^2, \) at \( p = (2, -1, -2, 5)^T \in G_\phi. \)

**Tangent Spaces for Image sets.**

7. Let the surface \( S \) be given parametrically as

\( S = \{ F(u) : u \in U \text{ and } JF(u) \text{ is of full-rank} \}, \)

of a \( C^1 \)-function \( F : U \subseteq \mathbb{R}^r \rightarrow \mathbb{R}^n. \) Let \( p \in S, \) so \( p = F(q) \) for some \( q \in U, \) at which the Jacobian matrix \( JF(q) \) is of full rank. Prove that

\[ \{ JF(q)w : w \in \mathbb{R}^r \} \subseteq T_p(S), \]

where \( T_p(S) \) is the Tangent Space to \( S \) at \( p. \)

**Hint** Look at the definition of \( T_p(S) \) and consider \( \gamma(t) = F(q + tw). \)

**Note** this is the easier half of a major result in the course that says there is equality in (1). The result is proven in the lectures when \( F \) is the image of a graph of some function \( \phi. \) The proof of equality is in the appendices to the notes and requires noting that a surface given parametrically is locally a graph. The complication comes from the fact that the result for graphs will give the Tangent Plane in terms of \( \phi \) and not the function \( F. \)

8. In each case, find parametric equations for the Tangent Plane passing through the point \( F(q) \) on the parametric surfaces given by the following functions.

i. \( F((x, y)^T) = (x^2 + y^2, xy, 2x - 3y)^T, \quad \text{at } q = (1, 2)^T, \)
\[ F((x, y)^T) = (xy^2, x^2 + y, x^3 - y^2, y^3)^T, \quad \text{at } q = (-1, 2)^T, \]

\[ F(t) = (\cos t, \sin t, t)^T \quad \text{at } q = 3\pi. \]

9 Find parametric equation for the Tangent Plane passing through the point \( F(q) \) on the parametric surface given by \( F(x) = (yz, xz, xy, xyz)^T \), for \( x = (x, y, z)^T \) at \( q = (1, -1, 2)^T \).

**Tangent Planes to level sets.**

10. Let the surface \( S \) be given as a level set
\[ S = \{ x \in U : f(x) = 0 \text{ and } Jf(x) \text{ is of full-rank} \}, \]
for some \( C^1 \)-function \( f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m \). Let \( p \in S \). Prove that
\[ T_p(S) \subseteq \{ x \in \mathbb{R}^n : Jf(p)x = 0 \}. \quad (2) \]

**Note** this is the (easier half) of a fundamental result in the course which states that we have equality in \( (2) \). See the appendices for a proof of the general result. It proceeds by using the Implicit Function Theorem to say that \( S \) is, locally near \( p \), a graph of some function \( \phi \) and a proof of \( (2) \) is given in the notes for a graph. The complication comes from the fact that the result for graphs will give the Tangent Plane in terms of \( \phi \) and not the function \( f \).

11. For each of the following level sets find the tangent plane to the surface at the given point \( p \) and give your answer as a level set.

i. \( (x, y, z)^T \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 14 \) with \( p = (2, 1, -3)^T \),

ii. \( (x, y, z)^T \in \mathbb{R}^3 : \)
\[ x^2 + 3y^2 + 2z^2 = 9, \]
\[ xyz = -2, \]

with \( p = (2, -1, 1)^T \),
iii. \((x, y, u, v) \in \mathbb{R}^4:\)
\[
\begin{align*}
  x^3 - 3yu + u^2 + 2xv &= 12 \\
  xv^2 + 2y^2 - 3u^2 - 3yv &= -3.
\end{align*}
\]
with \(p = (1, 2, -1, 2)^T\),

12. Return to your answers of Question 11 and write them as graphs instead of level sets. Then give a basis for the Tangent Space.

13. Return to Question 13 on Sheet 6. You were asked to show, by using the Implicit Function Theorem, that the following equations
\[
\begin{align*}
  x^2 + y^2 + 2uv &= 4 \\
  x^3 + y^3 + u^3 - v^3 &= 0,
\end{align*}
\]
determine \(u\) and \(v\) as functions of \(x\) and \(y\) for \((x, y)^T\) in an open subset of \(\mathbb{R}^2\) containing the point \(q = (-1, 1)^T \in \mathbb{R}^2\). The implicit function theorem is an existence result, it does not say what \(u\) and \(v\) are as functions of \(x\) and \(y\). Nonetheless it is possible to find their partial derivatives and you were asked to do this. The answer was
\[
\frac{\partial u}{\partial x}(q) = 0, \quad \frac{\partial v}{\partial x}(q) = 1, \quad \frac{\partial u}{\partial y}(q) = -1 \quad \text{and} \quad \frac{\partial v}{\partial y}(q) = 0.
\]
Use these partial derivatives to find a basis for the tangent space at \(p = (-1, 1, 1, 1)^T\).

14. Let \(S (u) = (\cos u \sin v, \sin u \sin v, \cos v)^T\), where \(u = (u, v)^T\), with \(0 \leq v \leq \pi, 0 \leq u \leq 2\pi\). This is the surface of the unit ball in \(\mathbb{R}^3\) in standard spherical coordinates.

i. Show that the tangent space of \(S\) at \(q = (\pi, \pi/2)^T\) is \(T_pS = \text{Span}(\mathbf{e}_2, \mathbf{e}_3)\), where \(p = S(q)\).

ii. Determine also the tangent space at \(q = (0, \pi/4)^T\).

iii. a. Let \(\mathbf{w} = (1, 2, -1)^T/\sqrt{6}\). Show that \(\mathbf{w} \in T_pS\) where \(p = S((0, \pi/4)^T)\).
b. But the definition of $T_pS$ is that $w \in T_pS$ only if there exists a curve $a : I \to S$ such that $\alpha(0) = p$ and $\alpha'(0) = w$. Find $\alpha$ in this case.

**Hint** In the notes we prove that $T_p(S) = \{ JF(q)x \}$ when $S = \text{Im} F$. Look at that proof which constructs a curve within a surface.
Additional Questions

15. For additional practice For each function, find the Best Affine Approximation at the given point.

i. \( f((s, t)^T) = (t \cos s, t \sin s, t, q = (\pi/2, 2)^T, \)

ii. \( f((s, t)^T) = (t^2 \cos s, t^2, t^2 \sin s, q = (0, 1)^T, \)

16 Complete the proof of the result stated in lectures:

Proposition i. \( P \) is a plane of dimension \( r \) in \( \mathbb{R}^n \) iff there exists \( p \in \mathbb{R}^n \) and a \( n \times r \) full-rank matrix \( M \) such that

\[ P = \{ p + Mt : t \in \mathbb{R}^r \}. \]

ii. \( P \) is a plane of dimension \( r \) in \( \mathbb{R}^n \) iff there exists \( p \in \mathbb{R}^n \) and an \( (n - r) \times n \) full-rank matrix \( N \) such that

\[ P = \{ x \in \mathbb{R}^n : N(x - p) = 0 \}. \]

Hint Remember the definition of a plane as an affine set. Also a vector space \( V \) of dimension \( r \) has a basis of \( r \) vectors which, by the Gram-Schmidt method, can be expanded to a basis of \( \mathbb{R}^n \). Then a vector is in \( V \) iff it is orthogonal to all the additional vectors in this basis of \( \mathbb{R}^n \).

17 In each of the following examples, find both a basis for the Tangent Space and the equation of the Tangent Plane to the graph of \( \phi \) for the given point:

i. \( \phi(x) = x(x + y) \), at \( q = (2, -1)^T \in \mathbb{R}^2, \)

ii. \( \phi(x) = (x - 1)^2 + y^2 \) at \( q = (0, 2)^T \in \mathbb{R}^2, \)

iii. \( \phi(x) = \sin(xy^2z^3) \) at \( q = (\pi, 1, -1)^T. \)

iv. \( \phi(x) = \begin{pmatrix} xy^2 \\ x^2y \end{pmatrix} \)

at \( q = (2, -3)^T \) and then again at \( q = (2, 1)^T, \)

v. \( \phi(x) = \begin{pmatrix} xy \\ yz \end{pmatrix} \)
at \( q = (1, -1, 2)^T \).

**Hint** Most of these functions have appeared in previous questions. It may save time to quote the results already proved.

18 Assume \( f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m \) is a \( C^1 \)-function on \( U \). Assume that at \( a \in U \) the Jacobian matrix \( Jf(a) \) is of full-rank. Prove that there exists an open set \( A : a \in A \subseteq U \) such that \( Jf(x) \) is of full-rank for all \( x \in A \).

19. Find the Tangent plane to the graph of

\[
\phi(x) = \frac{x^3 - y^3 + 1}{(x + y)^4 + 1},
\]

where \( x = (x, y)^T \in \mathbb{R}^2 \), at the point \( (2, -1, 5)^T \) on the graph.

**Hint** Multiply up before you differentiate.

20 Find the Tangent plane to the graph of

\[
\phi(x) = \frac{x^2y + 2xy^2}{1 + x^2 + y^2}
\]

where \( x = (x, y)^T \in \mathbb{R}^2 \), at the point \( q = (1, 2)^T \).

21 Let \( C \subseteq \mathbb{R}^3 \) be the level set

\[
\begin{align*}
x^2z^3 - x^3z^2 &= 0, \\
x^2y + xy^3 &= 2.
\end{align*}
\]

Show that in some neighbourhood of \( p = (1, 1, 1)^T \), \( C \) is a curve which can be parametrized by \( g(x) = (x, g_1(x), g_2(x)) \) for differentiable functions \( g_1 \) and \( g_2 \).

Find a parametrization of the Tangent Line to \( C \) at \( p \).

22 Find the Tangent Plane to the surface

\[
\begin{align*}
x^3 - y^3 + xv + uv &= 0, \\
xu^2 + yv^2 &= 0.
\end{align*}
\]
where \((x, y, u, v)^T \in \mathbb{R}^4\), at \(p = (-1, 1, -1, -1)^T\). Give your answer as a level set, and also as a graph. Find a basis for the Tangent Space to the surface at \(p\).

23. Find parametric equations for the tangent plane passing through the given point \(F(q)\) on the parametric surfaces given by
   i. \(F((x, y)^T) = (x^2 + y^2, xy, 2x - 3y)^T\) at \(q = (1, 1)^T\).
   ii. \(F((s, t)^T) = (t \cos s, t \sin s, t)^T\), \(q = (\pi/2, 2)^T\).
   iii. \(F((s, t)^T) = (t^2 \cos s, t^2, t^2 \sin s)^T\), \(q = (0, 1)^T\).

24. Define the function \(f : \mathbb{R}^2 \to \mathbb{R}^2\) by \(f((u, v)^T) = (u^3 + uv + v^3, u^2 - v^2)^T\). Show that \(f\) locally invertible at \(a = (1, 1)^T\).

   What is the Best Affine Approximation to the inverse function near \(b = f(a) = (3, 0)^T\)?

   What approximation does this give to \(f^{-1}(b')\) where \(b' = (3.1, -0.2)^T\)?

25. Find the tangent planes at the points \(p_1 = (1/\sqrt{2}, 1/4, 1/4)\) and \(p_2 = (\sqrt{3}/2, 0, 1/4)\) on the ellipsoid \(x^2 + 4y^2 + 4z^2 = 1\).

   Find the line of intersection of these two planes.

26. i. Find the Tangent Plane to the surface \(z = xe^y\) at the point \(p = (1, 0, 1)^T\) on the surface.

   ii. The surfaces \(x^2 + y^2 - z = 1\) and \(x + y + z = 5\) intersect in a curve \(\Gamma\). Find the equation in parametric form of the tangent line to \(\Gamma\) at the point \((1, 2, 2)^T\).

27. i. Consider the surface \(S = \{(x, y, z)^T \in \mathbb{R}^3 : xy = z\}\). Let \(p = (A, B, C)^T\) be a generic point of \(S\). Find the Tangent Plane at \(p\).

   ii. Show that the intersection of the Tangent Plane with \(S\) consists of two straight lines.