MATH20101 Real Analysis 2017-18

Exam 2017-18, Solutions and commonly seen problems

A1

(i) Prove, by verifying the ε - δ definition, that

$$\lim_{x\to 2}\left(x^3-3x^2+6\right)=2.$$

(ii) Prove the *Product Rule for Limits*: Assume that f and g are real valued functions defined on a deleted neighbourhood of $a\in\mathbb{R}$. Further assume that $\lim_{x\to a} f(x)=L$ and $\lim_{x\to a} g(x)=M$. Prove that

$$\lim_{x\to a} f(x)\,g(x)=LM.$$

(You may assume that if $\lim_{x\to a} h(x)=H$ then $|h(x)|<|H|+1$ in some deleted neighbourhood of a.)

(iii) Using the limit laws evaluate

$$\lim_{x\to -1}\frac{x^3+1}{x^3-4x^2-2x+3}$$

Solution

(i) *Rough work* Assume $0<|x-2|<\delta$, where $\delta>0$ is to be found. Consider

$$\begin{aligned}|f(x)-L| &= \left|\left(x^3-3x^2+6\right)-2\right|\\ &= \left|x^3-3x^2+4\right|\\ &= |x-2|\left|x^2-x-2\right|\\ &< \delta\left|x^2-x-2\right|.\end{aligned}$$

Assume $\delta\le 1$ when $|x-2|<\delta\le 1$ expands out as $1<x<3$. Then, by the triangle inequality,

$$\left|x^2-x-2\right|\le |x|^2+|x|+2<3^2+3+2=14,$$

since $|x|<3$. Thus $|f(x)-L|<14\delta$ which we demand $\le\varepsilon$. So we are led to $\delta=\min(1,\varepsilon/14)$. *End of Rough work.*

Proof Let $\varepsilon > 0$ be given. Choose $\delta = \min(1, \varepsilon/14)$ and assume $0 < |x-1| < \delta$. For such x we have, as in the Rough Work, that

$$\left|\left(x^3 - 3x^2 + 6\right) - 2\right| < \delta \left|x^2 - x - 2\right|,$$

$1 < x < 3$ and $|x^2 - x - 2| \leq 14$. Thus

$$\left|\left(x^3 - 3x^2 + 6\right) - 2\right| < 14\delta \leq 14\left(\frac{\varepsilon}{14}\right) = \varepsilon.$$

Hence we have verified the definition of

$$\lim_{x\to 2}\left(x^3 - 3x^2 + 6\right) = 2.$$

Alternatively You can further factor $x^2 - x - 2 = (x+1)(x-2)$ so that

$$|f(x) - L| = |x-2|^2 |x+1| \leq \delta^2 |x+1|.$$

Again $\delta \leq 1$ implies $1 < x < 3$ and thus $|x+1| < 4$. So this time demand that $4\delta^2 \leq \varepsilon$ which leads to $\delta = \min(1, \sqrt{\varepsilon}/2)$.

[**6 marks**]

(ii) Start with

$$\begin{aligned}
|f(x)\,g(x) - LM| &= |f(x)\,g(x) - Lg(x) + Lg(x) - LM|, \\
&\qquad \text{"adding in zero"}, \\
&\leq |f(x)\,g(x) - Lg(x)| + |Lg(x) - LM| \\
&\qquad \text{by the triangle inequality}, \\
&= |f(x) - L|\,|g(x)| + |L|\,|g(x) - M|
\end{aligned}$$

By the assumption given in the question $\lim_{x\to a} g(x) = M$ means there exists $\delta_1 > 0$ such that

$$0 < |x-a| < \delta_1 \Rightarrow |g(x)| < |M| + 1. \qquad (1)$$

Let $\varepsilon > 0$ be given.

The definition of $\lim_{x\to a} g(x) = M$ means there exists $\delta_2 > 0$ such that, if $0 < |x-a| < \delta_2$ then

$$|g(x) - M| < \frac{\varepsilon}{2\left(|L| + 1\right)}, \qquad (2)$$

(where we have put a +1 in the denominator, $2(|L|+1)$, in case $L=0$).

From the definition of $\lim_{x\to a} f(x) = L$ there exists $\delta_3 > 0$ such that, if $0 < |x-a| < \delta_3$ then

$$|f(x) - L| < \frac{\varepsilon}{2(|M|+1)}. \tag{3}$$

Choose $\delta = \min(\delta_1, \delta_2, \delta_3) > 0$. Assume $0 < |x-a| < \delta$. For such x all the three bounds (1), (2) and (3) hold.

Returning to result above

$$\begin{aligned} |f(x)\, g(x) - LM| &\le |f(x) - L|\,|g(x)| + |L|\,|g(x) - M| \\ &< \underbrace{\frac{\varepsilon}{2M'}}_{\text{by (3)}} \underset{\text{by (1)}}{M'} + |L| \underbrace{\frac{\varepsilon}{2(|L|+1)}}_{\text{by (2)}} \\ &= \left(\frac{1}{2} + \frac{1}{2} \times \underbrace{\frac{|L|}{(|L|+1)}}_{<1} \right) \varepsilon < \varepsilon. \end{aligned}$$

Thus we have verified the definition that $\lim_{x\to a} f(x)\, g(x) = LM$.

[9 marks]

(iii)

$$\begin{aligned} \lim_{x\to -1} \frac{x^3+1}{x^3 - 4x^2 - 2x + 3} &= \lim_{x\to -1} \frac{(x+1)(x^2 - x + 1)}{(x+1)(x^2 - 5x + 3)} \\ &= \lim_{x\to -1} \frac{x^2 - x + 1}{x^2 - 5x + 3} \\ &= \frac{\lim_{x\to -1}(x^2 - x + 1)}{\lim_{x\to -1}(x^2 - 5x + 3)} \quad \text{by Quotient Rule}^* \\ &= \frac{3}{9} = \frac{1}{3}. \end{aligned}$$

*The Quotient Law is applicable since both limits $\lim_{x\to -1}(x^2 - x + 1)$ and $\lim_{x\to -1}(x^2 - 5x + 3)$ exist with the second one non-zero.

Alternatively Though I had not wanted students to use L'Hôpital's Rule (why use differentiation when it is not necessary and it is also not a Limit

Law as described in my notes) so many students used it I allowed it. Since

$$\lim_{x\to -1}\left(x^3+1\right) = \lim_{x\to -1}\left(x^3-4x^2-2x+3\right) = 0$$

L'Hôpital's Rule gives

$$\lim_{x\to -1}\frac{x^3+1}{x^3-4x^2-2x+3} = \lim_{x\to -1}\frac{3x^2}{3x^2-8x-2} = \frac{3}{3+8-2} = \frac{1}{3}.$$

[5 marks]

Commonly seen errors.

i. Since we are looking at limits we cannot forget the "0 <" in $0 < |x-1|$. There were incorrect application of the triangle inequality, e.g. $|x^2-x-2| \leq |x|^2-|x|-2$. Note that $x<3$ in this (false) upper bound gives 4, which **is** an allowable upper bound but a false proof of a correct result will not get the marks.

Similarly, some students started from $1<x<3$ and just plugged the end points of this interval, 1 and 3, into x^2-x-2 claiming that the largest value, of 4 when $x=3$, was an upper bound for the quadratic. There are many examples in the problem sheets where the maximum of a quadratic over an interval does not occur at an end point so a proof of this claim was required.

ii In general students could either remember this proof or they could not. Small errors were incorrectly placing the "Let $\varepsilon>0$ be given" or again forgetting the "0 <" in $0<|x-a|<\delta$. Also, you had to justify **why** the δ_1, δ_2 and δ_3 exist.

iii It was important to tell me what Rule was being used. A surprising number of students made simple errors, such as

$$\lim_{x\to -1}\left(x^2-x+1\right) = 1-1+1 = 1,$$

i.e. not noticing the double negative in the middle.

A2

(i) Show, by verifying the definition, that

$$g(x) = \frac{x^2}{1+x}$$

is differentiable on $\mathbb{R} \setminus \{-1\}$ and find its derivative.

(ii) (a) State carefully *Rolle's Theorem.*

(b) State carefully the *Mean Value Theorem.*

(c) Deduce the Mean Value Theorem from Rolle's Theorem.

(iii) Prove that

$$\ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$$

for $x > 0$.

Solution

(i) Let $a \neq -1$ be given. Consider

$$\begin{aligned}
\lim_{x \to a} \frac{\frac{x^2}{(1+x)} - \frac{a^2}{(1+a)}}{x-a} &= \lim_{x \to a} \frac{x^2(1+a) - a^2(1+x)}{(x-a)(1+x)(1+a)} \\
&= \lim_{x \to a} \frac{x^2 - a^2 + x^2 a - a^2 x}{(x-a)(1+x)(1+a)} \\
&= \lim_{x \to a} \frac{(x+a)(x-a) + xa(x-a)}{(x-a)(1+x^2)(1+a^2)} \\
&= \lim_{x \to a} \frac{x+a+ax}{(1+x)(1+a)} \\
&= \frac{a^2+2a}{(1+a)^2},
\end{aligned}$$

by the Quotient Rule, allowable since $\lim_{x \to a}(x+a+ax)$ exists and

$$\lim_{x \to a} \left(1+x^2\right)\left(1+a^2\right) = \left(1+a^2\right)^2 \neq 0$$

for $a \neq -1$.

The limit exists so $x^2/(1+x)$ is differentiable at a.

True for all $a \in \mathbb{R}\setminus\{-1\}$ means it is differentiable on $\mathbb{R}\setminus\{-1\}$ with derivative $(x^2+2x)/(1+x)^2$.

[5 marks]

(ii) (a) **Rolle's Theorem** states that if a function f is differentiable on the open interval (a,b), continuous on the closed interval $[a,b]$ and $f(a)=f(b)$ then there exists $c : a<c<b$ such that $f'(c)=0$.

[2 marks]

(b) **The Mean Value Theorem** states that if a function f is differentiable on the open interval (a,b) and continuous on the closed interval $[a,b]$ then there exists $c : a<c<b$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}. \tag{4}$$

[2 marks]

(c) Let f be a function differentiable on the open interval (a,b) and continuous on the closed interval $[a,b]$. Define $F(x)=f(x)-kx$ where k is chosen such that $F(a)=F(b)$, i.e.

$$k = \frac{f(b)-f(a)}{b-a}.$$

Apply Rolle's Theorem to F to find $c \in (a,b) : F'(c)=0$. That is $f'(c)-k=0$, which gives the required result (4).

[6 marks]

(iii) Define

$$f(t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \ln(1+t),$$

for $t \geq 0$. Note that

$$f'(t) = 1 - t + t^2 - \frac{1}{1+t} = \frac{1+t-t-t^2+t^2+t^3-1}{1+t} = \frac{t^3}{1+t}.$$

In particular $f'(t)>0$ for all $t>0$.

Given $x>0$ apply the Mean Value Theorem to f on the interval $[0,x]$ to find $c : 0<c<x$ for which

$$f(x)-f(0) = f'(c)\,x > 0.$$

That is $f(x) > f(0) = 0$. This rearranges to required result.

[5 marks]

Commonly seen errors.

i. Many students misread their own work and cancelled $x^2 a$ with $a^2 x$. This led to a claimed derivative of $2a/(1+a)^2$; yet you should never get the wrong answer since you know from School days how to differentiate a quotient.

Many students *started* their answer writing

$$g'(a) = \lim_{x \to a} \frac{\frac{x^2}{(1+x)} - \frac{a^2}{(1+a)}}{x - a}, \tag{5}$$

but this is only true when the limit *exists*, and this is unknown at the start of the solution. A phrase similar to "*The limit exists so* $x^2/(1+x)$ *is differentiable at* a." is required later in the solution and then you can write (5).

ii.b. Too many students gave me **Cauchy's** Mean Value Theorem. I accepted this but there are more conditions, including $g'(x) \neq 0$ **for all** $x \in (a, b)$, which many students forgot.

ii.c. I accepted the proof of Cauchy's Mean Value Theorem. Many students started by saying "Define $F(x) = f(x) - kx$ such that $F(a) = F(b)$". I asked the obvious question 'what is k?'

iii. I did not accept "$f'(t) > 0$ for all $t > 0$ implies f is strictly increasing" This claim requires a proof, which is the essence of this question. The proof requires the Mean Value Theorem which is why the Mean Value Theorem is the subject of the earlier part of the question.

You **cannot** answer part iii by calculating the Taylor Series and discarding the powers greater than 3. You **can** answer the question by using Taylor's Theorem with Lagrange's form of the error, though only two students attempted this method.

A3

(i) State the ε - δ definition that $h : \mathbb{R} \to \mathbb{R}$ is *continuous* at $a \in \mathbb{R}$.

(ii) Assume that g is defined on a deleted neighbourhood of $a \in \mathbb{R}$ and that $\lim_{x\to a} g(x) = L$ exists. Assume that f is defined on a neighbourhood of L and is continuous at L. Prove that

$$\lim_{x\to a} f(g(x)) = f(L)$$

Hint. Consider f first.

(iii) Calculate the Taylor polynomial

$$T_{6,0}\left((1+x)\cos^2 x\right).$$

Solution

i. $h : \mathbb{R} \to \mathbb{R}$ is *continuous* at $a \in \mathbb{R}$ iff

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x : |x-a| < \delta \Rightarrow |h(x) - h(a)| < \varepsilon.$$

[2 **marks**]

ii. Let $\varepsilon > 0$ be given. Since f is continuous at L there exists $\delta_1 > 0$ such that,

$$|y - L| < \delta_1 \Rightarrow |f(y) - f(L)| < \varepsilon. \quad (6)$$

Choose $\varepsilon = \delta_1$ in the definition of $\lim_{x\to a} g(x) = L$ to find $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_2 \Rightarrow |g(x) - L| < \delta_1. \quad (7)$$

Combine (6) and (7) (using $g(x)$ in place of y in (6)) to get

$$\begin{aligned} 0 < |x-a| < \delta_2 \quad &\Rightarrow \quad |g(x) - L| < \delta_1 \\ &\Rightarrow \quad |f(g(x)) - f(L)| < \varepsilon. \end{aligned}$$

Thus we have verified the definition that

$$\lim_{x\to a} f(g(x)) = f(L).$$

[6 **marks**]

(iii) If $f(x) = (1+x)\cos^2 x$ then

$$f^{(1)}(x) = \cos^2 x - 2(x+1)\cos x \sin x = \cos^2 x - (x+1)\sin 2x.$$

Continuing,

$$\begin{aligned}
f^{(2)}(x) &= -2\sin x \cos x - \sin 2x - 2(x+1)\cos 2x \\
&= -2\sin 2x - 2(x+1)\cos 2x, \\
f^{(3)}(x) &= -4\cos 2x - 2\cos 2x + 4(x+1)\sin 2x \\
&= -6\cos 2x + 4(x+1)\sin 2x, \\
f^{(4)}(x) &= 12\sin 2x + 4\sin 2x + 8(x+1)\cos 2x \\
&= 16\sin 2x + 8(x+1)\cos 2x, \\
f^{(5)}(x) &= 40\cos 2x - 16(x+1)\sin 2x, \\
f^{(6)}(x) &= -96\sin 2x - 32(x+1)\cos 2x.
\end{aligned}$$

Thus $f(0) = 1$, $f^{(1)}(0) = 1$, $f^{(2)}(0) = -2$, $f^{(3)}(0) = -6$, $f^{(4)}(0) = 8$, $f^{(5)}(0) = 40$ and $f^{(6)}(0) = -32$. Hence

$$\begin{aligned}
T_{6,0}\left((1+x)\cos^2 x\right) &= 1 + x - 2\frac{x^2}{2!} - 6\frac{x^3}{3!} + 8\frac{x^4}{4!} + 40\frac{x^5}{5!} - 32\frac{x^6}{6!} \\
&= 1 + x - x^2 - x^3 + \frac{1}{3}x^4 + \frac{1}{3}x^5 - \frac{2}{45}x^6.
\end{aligned}$$

[12 marks]

Commonly seen errors.

ii. Many students wrote ‘Choose $\delta_1 = \varepsilon$’ instead of ‘Choose $\varepsilon = \delta_1'$. Other students looked at g first instead of f.

iii. There were many problems with differentiation, forgetting the negative sign on differentiating $\cos x$; forgetting the 2 on differentiating $\cos^2 x$; forgetting how to differentiate a product such as $(x+1)\cos x \sin x$. This all indicated a lack of practice.

Not so much an error, but **not** observing that

$$2\cos x \sin x = \sin 2x$$

makes the calculations longer (and more prone to error.) So, if you don't simplify the last term in

$$f^{(1)}(x) = \cos^2 x - 2(x+1)\cos x \sin x,$$

the next derivative is

$$\begin{aligned} f^{(2)}(x) &= -2\sin x \cos x - 2\cos x \sin x + 2(x+1)\sin^2 x - 2(x+1)\cos^2 x \\ &= -4\cos x \sin x + 2(x+1)\sin^2 x - 2(x+1)\cos^2 x. \end{aligned}$$

Then

$$\begin{aligned} f^{(3)}(x) &= 4\sin^2 x - 4\cos^2 x + 2\sin^2 x - 2\cos^2 x \\ &\quad +4(x+1)\sin x \cos x + 4(x+1)\cos x \sin x \\ &= 6\sin^2 x - 6\cos^2 x + 8(x+1)\sin x \cos x. \end{aligned}$$

At each step we have more terms.

Note, writing

$$\frac{f(x)}{(1+x)} = \cos^2 x,$$

does not give any advantage. My guiding principle is that I do **not** like fractions; differentiating quotients always leads to complicated expressions.

A4

(i) Assume f is a bounded function on the interval $[a, b]$.

(a) Define what is meant by saying that $\mathcal{P}$ *is a partition of* $[a, b]$.

(b) Define the

$$\textit{Upper integral} \ \overline{\int_a^b} f \quad \text{and} \quad \textit{Lower integral} \ \underline{\int_a^b} f,$$

not forgetting to define all the terms you use.

(c) Prove that the lower and upper sums satisfy

$$L(\mathcal{Q}, f) \leq U(\mathcal{R}, f)$$

for any partitions $\mathcal{Q}$ and $\mathcal{R}$ of $[a, b]$.
(You may assume that $L(\mathcal{P}, f) \leq U(\mathcal{P}, f)$ for any partition $\mathcal{P}$ while

$$L(\mathcal{P}, f) \leq L(\mathcal{D}, f) \quad \text{and} \quad U(\mathcal{D}, f) \leq U(\mathcal{P}, f)$$

whenever $\mathcal{P} \subseteq \mathcal{D}$.)

(d) Deduce that

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

(ii) Let $f : [2, 8] \to \mathbb{R}, x \mapsto 1/x^3$ and, for every $n \geq 1$, define the partition

$$\mathcal{Q}_n = \left\{ 2\eta^i : 0 \leq i \leq n \right\},$$

where $\eta^n = 4$.

(a) Show that

$$L(\mathcal{Q}_n, f) = \frac{15}{64\eta\,(1 + \eta)}.$$

(You may assume that $\sum_{i=1}^n x^i = x\,(1 - x^n)\,/\,(1 - x)$.)

(b) Prove, by verifying the definition, that f is integrable over $[2, 8]$ and find the value of the integral.

(You may assume that $U(\mathcal{Q}_n, f) = 15\eta^2/64\,(1 + \eta)$.)

Solution A4

(i) (a) A partition of an interval $[a, b]$ is a set $\mathcal{P} = \{x_0, x_1, ...x_n\}$ where $a = x_0 < x_1 < x_2 < ... < x_n = b$. **[1 mark]**

(b) The ***Upper integral*** is

$$\overline{\int_a^b} f = \text{glb}\left\{U(\mathcal{P}, f) : \mathcal{P} \text{ a partition of } [a, b]\right\},$$

and the ***Lower integral*** is

$$\underline{\int_a^b} f = \text{lub}\left\{L(\mathcal{P}, f) : \mathcal{P} \text{ a partition of } [a, b]\right\}.$$

Here

$$U(\mathcal{P}, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) \quad \text{where} \quad M_i = \text{lub}\left\{f(x) : x \in [x_{i-1}, x_i]\right\},$$

and

$$L(\mathcal{P}, f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) \quad \text{where} \quad m_i = \text{glb}\left\{f(x) : x \in [x_{i-1}, x_i]\right\},$$

where $\mathcal{P} = \{x_i\}_{0 \le i \le n}$.

[4 marks]

(c) Given two partitions $\mathcal{Q}$ and $\mathcal{R}$ then $\mathcal{Q} \subseteq \mathcal{Q} \cup \mathcal{R}$ and $\mathcal{R} \subseteq \mathcal{Q} \cup \mathcal{R}$. So

$$\begin{aligned} L(\mathcal{Q}, f) &\le L(\mathcal{Q} \cup \mathcal{R}, f) \quad \text{by assumption,} \\ &\le U(\mathcal{Q} \cup \mathcal{R}, f) \quad \text{by assumption,} \\ &\le U(\mathcal{R}, f) \quad \text{by assumption again.} \end{aligned}$$

[3 marks]

(d) Fix $\mathcal{R}$ and vary $\mathcal{Q}$. We thus see that $U(\mathcal{R}, f)$ is *an* upper bound for $\{L(\mathcal{Q}, f) : \mathcal{Q}\}$. Yet $\underline{\int_a^b} f$ is the *least* of all upper bounds and thus less than the upper bound $U(\mathcal{R}, f)$, i.e.

$$U(\mathcal{R}, f) \ge \underline{\int_a^b} f.$$

Now vary $\mathcal{R}$ and see that $\underline{\int_a^b} f$ is *a* lower bound for $\{U(\mathcal{R}, f) : \mathcal{R}\}$. Yet $\overline{\int_a^b} f$ is the *greatest* of all lower bounds, so greater than the lower bound $\underline{\int_a^b} f$, i.e.

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

as required. **[3 marks]**

(ii) (a) Let $f : [2, 8] \to \mathbb{R}, x \mapsto 1/x^3$. With the partition $\mathcal{Q}_n = \{2\eta^i : 0 \leq i \leq n\}$ the sub-intervals are $[x_{i-1}, x_i] = [2\eta^{i-1}, 2\eta^i]$ which have width $2\left(\eta^i - \eta^{i-1}\right)$. The function f is decreasing so it is minimum on $[2\eta^{i-1}, 2\eta^i]$ at $2\eta^i$. Thus

$$\begin{aligned} L(\mathcal{Q}_n, f) &= \sum_{i=1}^{n} 2\left(\eta^i - \eta^{i-1}\right) \frac{1}{(2\eta^i)^3} = \frac{1}{2^2}\left(1 - \eta^{-1}\right) \sum_{i=1}^{n} \frac{1}{(\eta^i)^2} \\ &= \frac{1}{4}\left(1 - \eta^{-1}\right) \sum_{i=1}^{n} \left(\frac{1}{\eta^2}\right)^i \\ &= \frac{1}{4}\left(1 - \eta^{-1}\right) \frac{1}{\eta^2} \frac{1 - \eta^{-2n}}{1 - \eta^{-2}} \\ &= \frac{1}{4}\left(1 - \eta^{-1}\right) \frac{1}{\eta^2} \frac{1 - (1/4)^2}{(1 - \eta^{-1})(1 + \eta^{-1})} \end{aligned}$$

having used $\eta^n = 4$. Thus

$$L(\mathcal{Q}_n, f) = \frac{15}{64\eta(1 + \eta)}.$$

[7 marks]

(b) For every $n \geq 1$ we get

$$\frac{15}{64\eta(1+\eta)} = L(\mathcal{Q}_n, f) \leq \underline{\int_2^6} f \leq \overline{\int_2^6} f \leq U(\mathcal{Q}_n, f) = \frac{15\eta^2}{64(1+\eta)}.$$

Let $n \to \infty$ when $\eta = 4^{1/n} \to 1$, to see that we must have equality in the centre, i.e. $\underline{\int_2^8} f = \overline{\int_2^8} f$, and so the function in integrable over $[2, 8]$. The common value, $15/128$, is therefore the value of the integral.

[2 marks]

Commonly seen errors.

i. Many students could not give the definition of a partition; forgetting it is basically a finite subset of the interval, with the end points of the interval in the set.

Students often interchanged lub and glb in the definitions; if you could recall some of the diagrams I drew in lectures and in the notes that might have helped. Without the correct definitions it was hard to give the required proof in part i.d.

ii. Many students gave (something looking like) the correct result

$$L(\mathcal{Q}_n, f) = \sum_{i=1}^{n} 2\left(\eta^i - \eta^{i-1}\right)\frac{1}{\left(2\eta^i\right)^3}$$

but then had problems summing this, even given the formula for summing a geometric series. This was a problem with algebraic manipulation and indicated a lack of practice.