# MATH10242 Sequences and Series: Solutions 8, for exercises for week 9 Tutorials

**Question 1:** Use partial fractions to find: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$.

**Solution:** Write as partial fractions,

$$\frac{1}{4n^2-1} = \frac{A}{2n+1} + \frac{B}{2n-1} = \frac{(2n-1)A + (2n+1)B}{4n^2-1}.$$

From this we get $A+B=0$ and $B-A=1$; thus $B=1/2$ and $A=-1/2$. Thus

$$\begin{aligned}\sum_{n=1}^{t} \frac{1}{4n^2-1} &= \sum_{n=1}^{t} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \frac{1}{2}\left(\left(\frac{1}{1}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{2t-1} - \frac{1}{2t+1}\right)\right).\end{aligned}$$

The intermediate terms cancel and we get

$$\sum_{n=1}^{t} \frac{1}{4n^2-1} = \frac{1}{2}\left(1 - \frac{1}{2t+1}\right).$$

[[More formally:

$$\begin{aligned}\sum_{n=1}^{t} \frac{1}{4n^2-1} &= \sum_{n=1}^{t} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &= \frac{1}{2}\left(\sum_{n=1}^{t} \frac{1}{2n-1} - \sum_{n=1}^{t} \frac{1}{2n+1}\right) \\ &= \frac{1}{2}\left(\sum_{n=1}^{t} \frac{1}{2n-1} - \sum_{n=2}^{t+1} \frac{1}{2n-1}\right),\end{aligned}$$

having changed the 'variable of summation' in the second sum. You could do this in two steps, change from $n$ to $m = n+1$, and then relabel $m$ as $n$. Then most terms of the two sums cancel out, leaving

$$\sum_{n=1}^{t} \frac{1}{4n^2-1} = \frac{1}{2}\left(\frac{1}{1} - \frac{1}{2t+1}\right)$$

as seen before.]]

Hence

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{2t+1}\right) = \frac{1}{2}.$$

As was mentioned in lectures, it is important in these questions to only do the rearranging of terms for a finite sum. Only after doing that can you take the limit.

**Question 2:** Test the series below for convergence or divergence using the tests indicated. For the Comparison Test you need to decide whether you are expecting convergence (in which case you need to find a convergent series $\sum b_n$ with $b_n \geq a_n$ for all $n$) or divergence (in which case you need to find a divergent series $\sum b_n$ with $b_n \leq a_n$ for all $n$).

(a) *Comparison Test*

$$(i)\ \sum_{n\geq 1} \frac{n+1}{n^2+2}, \quad (ii)\ \sum_{n\geq 1} \frac{3n^2+2}{n^4+4}, \quad (iii)\ \sum_{n\geq 1} \frac{1}{2^n+n^2}, \quad (iv)\ \sum_{n\geq 2} \frac{1}{\ln n}.$$

*(b) Ratio Test*

$$(v)\ \sum_{n\geq 1} \frac{n^3}{3^n}, \quad (vi)\ \sum_{n\geq 1} \frac{3^n}{n!}, \quad (vii)\ \sum_{n\geq 1} \frac{n^n}{n!}, \quad (viii)\ \sum_{n=1}^{\infty} \frac{n!}{n^n}, \quad (ix)\ \sum_{n=1}^{\infty} \frac{n+1}{n^2+2}.$$

[Remark: you may use that $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$.]

**Solutions:**

(a) **Comparison Test**

As with sequences, identify the fastest-growing terms and look for what simpler series this new one is "essentially like". Then try to make the comparison with that simpler series (at this point, although there's usually not so much choice in which series to compare to, there are generally many ways of making that comparison, so quite likely you won't have made the comparisons in exactly the way they are done here).

(i)

$$\frac{n+1}{n^2+2} \geq \frac{n}{n^2+2} = \frac{1}{n+2/n} \geq \frac{1}{n+2}.$$

But

$$\sum_{n\geq 1} \frac{1}{n+2} = \sum_{n\geq 3} \frac{1}{n}$$

diverges, so by the Comparison Test (CT)

$$\sum_{n\geq 1} \frac{n+1}{n^2+2}$$

also diverges.

(ii)

$$0 \leq \frac{3n^2+2}{n^4+4} \leq \frac{3n^2+2}{n^4} \leq \frac{3n^2+2n^2}{n^4} = \frac{5}{n^2}.$$

But $\sum_{n\geq 1} 1/n^2$ converges by 9.1.4 and hence so does $\sum_{n\geq 1} 5/n^2$ (see 9.1.5). So, by the CT our series

$$\sum_{n\geq 1} \frac{3n^2+2}{n^4+4}$$

converges.

(iii)
$$\frac{1}{2^n+n^2} \le \frac{1}{2^n}.$$
Since $\sum_{n\ge1} 1/2^n$ is a convergent Geometric Series (8.1.1), so does our series, by the CT. (Comparison with $\sum \frac{1}{n^2}$ also works.)

(iv) Since $0 \le \frac{1}{\ln(n)} \ge \frac{1}{n}$, $\sum_{n\ge2} \frac{1}{\log n}$ diverges by the CT and comparison with the Harmonic Series (Section 8.1).

**(b) Ratio Test** (Note that, since all the terms here are positive, we don't need the modulus signs.)

(v) For $\sum_{n\ge1} n^3/3^n$, we take $a_n = n^3/3^n$ and compute:
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3}{3^{n+1}}\,\frac{3^n}{n^3} = \frac{(n+1)^3}{n^3}\,\frac{1}{3} = \frac{(1+1/n)^3}{1}\,\frac{1}{3} \to \frac{1}{3} < 1$$
as $n\to\infty$. So, by the Ratio Test (RT), the series $\sum_{n\ge1} n^3/3^n$ converges.

(vi) Here $a_n = 3^n/n!$, so
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{(n+1)!}\,\frac{n!}{3^n} = \frac{3}{n+1} \to 0,$$
as $n\to\infty$. So, by the RT, the series $\sum_{n\ge1} 3^n/n!$ converges.

(vii) Here $a_n = n^n/n!$, so
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!}\,\frac{n!}{n^n} = \frac{(n+1)^n(n+1)}{(n+1)!}\,\frac{n!}{n^n} = \frac{(n+1)^n}{n^n} = (1+1/n)^n \to e,$$
as $n\to\infty$ (where we used the hint). As $e>1$ this implies that $\sum_{n\ge1} n^n/n!$ diverges.

(viii) $\sum_{n=1}^{\infty} n!/n^n$. By taking the reciprocal of Part (iii) we get
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{e} < 1$$
so the series converges.

(ix) $\sum_{n=1}^{\infty}(n+1)/(n^2+2)$. Of course, by part (i) we know this diverges, but the point of the question is to illustrate that the Ratio Test is not useful for comparing polynomials.

Here
$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+2}{(n+1)^2+2}\cdot\frac{n^2+2}{n+1} = \frac{(n+2)(n^2+2)}{(n^2+2n+2)(n+1)} = \frac{(1+\frac{2}{n})(1+\frac{2}{n^2})}{(1+\frac{2}{n}+\frac{2}{n^2})(1+\frac{1}{n})}.$$
Thus, by the AoL
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{(1+0)(1+0)}{(1+0)(1+0)} = 1.$$
So, you cannot draw any conclusion.

**Question 3:** (a) Prove that if $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty}(a_n + b_n)$ diverges.

(b) Suppose that $\sum_{n=1}^{\infty}(a_n + b_n)$ converges and that $a_n \geq 0$ and $b_n \geq 0$ for all $n \in \mathbb{N}$. Prove that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge.

(c) In part (b), why did we need $a_n \geq 0$ and $b_n \geq 0$?

**Solutions:** (a) Suppose, for a contradiction, that $\sum_{n\geq 1}(b_n + a_n)$ converges. Then so does $\sum_{n\geq 1} -a_n = (-1)\sum_{n\geq 1} a_n$ by 8.1.5(b). Thus so does

$$\sum_{n\geq 1} b_n = \sum_{n\geq 1}(b_n + a_n) + (-1)\sum_{n\geq 1} a_n$$

by 8.1.5(a). This is a contradiction and so $\sum_{n\geq 1}(b_n + a_n)$ diverges.

[[That is a slightly sneaky proof. You can also do it direct from the definition, though it would take longer.]]

(b) *First proof:* We use the boundedness theorem 9.1.1 twice. First, as $a_n + b_n \geq 0\ \forall\, n \in \mathbb{N}$, that theorem says that since $\sum(a_n + b_n)$ converges, the partial sums are bounded, say $\sum_1^N(a_n + b_n) \leq T$ for all $N \geq 1$. But as $b_n \geq 0$ we have $a_n \leq (a_n + b_n)\ \forall\, n \in \mathbb{N}$ we get $\sum_1^N a_n \leq T$ for all $N \geq 1$. So, by 9.1.1 again $\sum a_n$ converges.

*Second Proof*: Here is a slight generalisation that can be useful:

**Lemma.** *Suppose that $\sum_{n=1}^{\infty}(a_n + b_n)$ converges and there exists some $N \in \mathbb{N}$ such that $a_n \geq 0$ and $b_n \geq 0$ for all $n \geq N$. Then both $\sum_{n\geq 1} a_n$ and $\sum_{n\geq 1} b_n$ converge.*

*Proof.* In this case, we only worry about the sequences $\sum_{n\geq N} a_n$ and $\sum_{n\geq N}(a_n + b_n)$. By 9.1.3 (that is, Question 4(b)) $\sum_{n\geq N}(a_n + b_n)$ still converges. But now it is a sum of positive integers, so we can apply 9.1.1. So, let

$$u_n = (a_N + b_N) + (a_{N+1} + b_{N+1}) + \cdots (a_n + b_n)$$

for $n \geq N$. By 9.1.1, $(u_n)$ is bounded above. Since $b_n \geq 0$ for all $n \geq N$, we see that

$$s_n = a_N + a_{N+1} + \cdots a_n \leq u_n$$

and so it is also bounded.

Hence by 9.1.1 the series $\sum_{n\geq N} a_n$ is convergent. Hence so is $\sum_{n\geq 1} a_n$ by 9.1.3.

(b) As you might guess, the result fails if we allow some negative numbers. For example, take $a_n = 1$ for all $n$; so $\sum_{n\geq 1} 1$ certainly diverges. But if $b_n = -1$ for all $n$ then again $\sum_{n\geq 1} b_n$ diverges, but

$$\sum_{n\geq 1}(a_n + b_n) = \sum_{n\geq 1} 0 = 0$$

converges.

**Question 4:** (a) Prove Theorem 8.1.5(ii): Suppose that $\sum_{n=1}^{\infty} a_n = s$ and that $\lambda$ is any real number. Prove that the series $\sum_{n=1}^{\infty} \lambda a_n$ converges with sum $\lambda s$.
(b) Prove 9.1.3: Given $N \geq 1$ and a series $\sum_{n\geq 1} a_n$, then $\sum_{n\geq 1} a_n$ converges $\iff$ $\sum_{n\geq N} a_n$ converges.

**Solutions:** (a) We are given that $\sum_{n=1}^{\infty} a_n = s$ and that $\lambda \in \mathbb{R}$. Thus the partial sums $s_n = a_1 + \cdots + a_n$ have $\lim_{n\to\infty} s_n = s$. By the AoL for sequences $t_n = \lambda s_n$ has $\lim_{n\to\infty} t_n = \lambda s$. But, the $t_n = \lambda a_1 + \cdots + \lambda a_n$ are the partial sums for $\sum_{n=1}^{\infty} \lambda a_n$. Hence

$$\sum_{n=1}^{\infty} \lambda a_n \;=\; \lim_{n\to\infty} t_n \;=\; \lambda s.$$

(b) If you convert this into a question about sequences you will find it is really just 4.1.3. Here are the details.

Let $s_n = a_1 + \cdots a_n$ be the partial sums for $\sum_{n\geq 1} a_n$ and let $t_n = a_N + \cdots a_n$ be the partial sums for $\sum_{n\geq N} a_n$ (where I either just start the sequence at $n = N$ or declare $t_1 = t_2 = \cdots = t_{N-1} = 0$).

Set $X = a_1 + \cdots + a_{N-1}$. Then $t_n = s_n - X$ for all $n \geq N$. Thus $\lim_{n\to\infty} s_n = \ell \iff \lim_{n\to\infty} t_n = \ell - X$, as in 4.1.3. So, certainly $\lim_{n\to\infty} s_n$ exists $\iff$ $\lim_{n\to\infty} t_n$ exists.

**Question 5*:** (a) Suppose that $\{a_n, b_n : n \geq 1\}$ are all positive and that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \ell$$

exists. Prove that if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.
(b) What happens in (b) if we allow negative terms? [You might find this easier after next week's lectures.]

**Solutions:** (a) *First proof:* Taking $\epsilon = 1$ we can find $N$ such that, if $n \geq N$ then

$$\frac{a_n}{b_n} \leq \ell + 1.$$

Equivalently, $a_n \leq (\ell + 1)b_n$.
Now,

$$\begin{aligned}
\sum_{n\geq 1} b_n \text{ converges } &\Rightarrow \sum_{n\geq N} b_n \text{ converges} \\
&\Rightarrow \text{the sequence } t_n = b_N + \cdots + b_n \text{ is bounded for } n \geq N \text{ by 9.1.1} \\
&\Rightarrow (\ell+1)t_n = (\ell+1)b_N + \cdots + (\ell+1)b_n \text{ is bounded for } n \geq N \\
&\Rightarrow a_N + \cdots + a_n \text{ is bounded for } n \geq N \text{ since } a_j \leq (\ell+1)b_j \\
&\Rightarrow \sum_{n\geq N} a_n \text{ converges (by 9.1.1 again)} \\
&\Rightarrow \sum_{n\geq 1} a_n \text{ converges (by 9.1.3 again).}
\end{aligned}$$

*Alternative (and faster) proof, pointed out by a postgrad demonstrator:* Since the sequence $a_n/b_n$ is convergent, it is bounded above by $M$ say. Then, for every $n \geq 1$ we have $a_n/b_n \leq M$, that is $a_n \leq Mb_n$. Since $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} Mb_n$ and hence, by Comparison, so does $\sum_{n=1}^{\infty} a_n$.

(b)* Here you can't draw the same conclusion. It might be that $\sum b_n$ converges but $\sum a_n$ doesn't. For instance, set $b_n = (-1)^n/\sqrt{n}$ so, by the Alternating Series Test, $\sum b_n$ converges. Set $a_n = 1/n$. Now, the limit,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

exists, yet $\sum a_n$ diverges.