MATH10242 Sequences and Series: Solutions 7, to exercises for week 8 Tutorials

Question 1 i. Show that

$$\left(\frac{1}{n^3-2n+2}\right)_{n\geq 1}$$

is a subsequence of $(1/n)_{n\geq 1}$.
ii. Is

$$\left(\frac{2}{6^n}\right)_{n\geq 1}$$

a subsequence of $(1/n)_{n\geq 1}$? Justify your answer.

Solution i. Choose $k_n = n^3-2n+2$. Need show that $k_{n+1} > k_n$ for all $n \geq 1$. Consider

$$\begin{aligned} k_{n+1}-k_n &= (n+1)^3-2(n+1)+2-(n^3-2n+2) \\ &= n^3+3n^2+3n+1-2n-2+2-(n^3-2n+2) \\ &= 3n^2+3n-1 > 0 \end{aligned}$$

for all $n \geq 1$ as required.

ii. Yes. Choose

$$k_n = \frac{6^n}{2} = 3^n 2^{n-1}.$$

Obviously an increasing sequence.

Question 1 If $(b_n)_{n\geq 1}$ is a subsequence of $(a_n)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ a subsequence of $(b_n)_{n\geq 1}$ show that $(c_n)_{n\geq 1}$ is a subsequence of $(a_n)_{n\geq 1}$.

Solution By definition, $b_n = a_{k_n}$ for some strictly increasing sequence (k_n). Similarly, $c_n = b_{\ell_n}$ for some strictly increasing sequence (ℓ_n).

Then $c_n = b_{\ell_n} = a_{k_{\ell_n}}$. We need check that $(k_{\ell_n})_{n\geq 1}$ is a strictly increasing sequence. But $n+1 > n$ implies $\ell_{n+1} > \ell_n$ since $(\ell_n)_{n\geq 1}$ is a strictly incresing sequence. And $\ell_{n+1} > \ell_n$ implies $k_{\ell_{n+1}} > k_{\ell_n}$ since $(k_n)_{n\geq 1}$ is a strictly increasing sequence. hence $(k_{\ell_n})_{n\geq 1}$ is a strictly increasing sequence as required.

Question 3: Using L'Hôpital's Rule, or otherwise, find the limit of the sequences

(i) $\left(\frac{\ln(7n^{1/4}-2)}{\ln(n+1)}\right)_{n\geq 1}$ (ii) $\left(\frac{e^{e^n}}{e^n}\right)_{n\geq 1}$

(iii) $\left(\frac{1-e^{-n}}{2-e^{-2n}}\right)_{n\geq 1}$ (iv) $\left(\frac{1-e^{n}}{2-e^{2n}}\right)_{n\geq 1}$

Solutions: (i) If $f(x) = \ln(7x^{1/4}-2)$ and $g(x) = \ln(x+1)$, then

$$a_n = \frac{\ln(7n^{1/4}-2)}{\ln(n+1)} = \frac{f(n)}{g(n)}.$$

Clearly
$$\lim_{n\to\infty} f(n) = +\infty = \lim_{n\to\infty} g(n)$$
and $g(x) > 0$ for $x \geq 1$, so the hypotheses of L'Hôpital's rule are satisfied. Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)} = \lim_{n\to\infty} \frac{(\frac{7}{4}n^{-3/4})(7n^{1/4}-2)^{-1}}{(n+1)^{-1}} = \lim_{n\to\infty} \frac{7}{4} \frac{(n+1)}{(7n-2n^{3/4})}.$$

Now the fastest-growing term is n so by dividing top and bottom by n and using the AoL we get
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{7}{4} \frac{(n+1)}{(7n-2n^{3/4})} = \frac{7}{4} \cdot \frac{(1+0)}{(7-0)} = \frac{1}{4}.$$

(ii) Again $f(x) = e^{e^x} \to \infty$ and $g(x) = e^x \to \infty$ as $x \to \infty$ and g is positive. So we can apply L'Hôpital's rule to get

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)} = \lim_{n\to\infty} \frac{e^n e^{e^n}}{e^n} = \lim_{n\to\infty} e^{e^n} = \infty.$$

(iii) If you're not careful, you might argue as follows: By L'Hôpital's rule,

$$\lim_{n\to\infty} \frac{1-e^{-n}}{2-e^{-2n}} = \lim_{n\to\infty} \frac{e^{-n}}{2e^{-2n}} = \lim_{n\to\infty} \frac{e^n}{2} = +\infty.$$

But that's not correct - what's wrong with this argument? What is the correct value of the limit? See the last page for the answer.

(iv) Here, either by dividing top and bottom through by e^{2n} (or using L'Hôpital's rule, which is valid here) one sees that the sequence has limit zero.

Question 4: (i) Use L'Hôpital's Rule to show that $(\ln n)^2/n \to 0$ as $n \to \infty$.
(ii) Show by induction that for any $k \in \mathbb{N}$, $(\ln n)^k/n \to 0$ as $n \to \infty$.

Solution: (i) We take $f(x) = (\ln x)^2$ and $g(x) = x$; thus

$$a_n = \frac{(\ln n)^2}{n} = \frac{f(n)}{g(n)}.$$

Clearly
$$\lim_{n\to\infty} f(n) = +\infty = \lim_{n\to\infty} g(n)$$
and $g(x) > 0$ for $x > 0$. Thus the hypotheses of L'Hôpital's rule are satisfied. Thus,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)} = \lim_{n\to\infty} \frac{2\frac{\ln n}{n}}{1} = \lim_{n\to\infty} 2\frac{\ln n}{n}.$$

Now either by another application of L'Hôpital's rule or the notes (see 8.1.3), we get

$$\lim_{n\to\infty} a_n = 2 \lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

(ii) We show by induction that $(\ln n)^k/n \to 0$ as $n \to \infty$. We have already done the case $k = 1$ in the notes (and $k = 2$ above), so assume that the result holds for some integer $k \geq 2$. Then by L'Hôpital's rule (which does again apply!) with $f(x) = (\ln x)^{k+1}$ and $g(x) = x$ we get

$$\lim_{n\to\infty} \frac{(\ln n)^{k+1}}{n} = \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)} = \lim_{n\to\infty} \frac{\frac{(k+1)(\ln n)^k}{n}}{1} = (k+1) \lim_{n\to\infty} \frac{(\ln n)^k}{n}.$$

But this final limit is zero by the inductive hypothesis. Hence $\lim_{n\to\infty}(\ln(n))^{k+1}/n = 0$ and the inductive step is complete. In other words,

$$\lim_{n\to\infty} \frac{(\ln n)^m}{n} = 0$$

for all integers m.

The observant reader may have noticed that we did not need to prove part (i) since it is contained in part (ii).

Question 5: (i) Using the formula

$$(x-y) = \frac{(x-y)(x^2+xy+y^2)}{(x^2+xy+y^2)} = \frac{(x^3-y^3)}{(x^2+xy+y^2)}$$

or otherwise, find

$$\lim_{n\to\infty} \sqrt[3]{n^3+n^2} - n.$$

(ii) Show that $[\sqrt[3]{n^3+n^2}\,] = n$.
(iii) Using subsequences show that $[\sqrt[3]{n}\,] - \sqrt[3]{n}$ does not have a limit.

Solution: (i) Substituting $x = \sqrt[3]{n^3+n^2}$ and $y = n$ into $(x-y) = \dfrac{(x^3-y^3)}{(x^2+xy+y^2)}$ gives

$$\sqrt[3]{n^3+n^2} - n = \frac{((n^3+n^2)-n^3)}{((n^3+n^2)^{2/3} + n(n^3+n^2)^{1/3} + n^2)} = \frac{n^2}{((n^3+n^2)^{2/3} + n(n^3+n^2)^{1/3} + n^2)}$$

$$= \frac{1}{((1+n^{-1})^{2/3} + (1+n^{-1})^{1/3} + 1)} \to \frac{1}{3} \quad as\ n \to \infty.$$

(ii) It suffices to prove that $n \leq \sqrt[3]{n^3+n^2} < n+1$. The left hand inequality is obvious while the right hand inequality is equivalent to $(n^3+n^2) < (n+1)^3 = n^3+3n^2+3n+1$, which is certainly true.

(iii) We use subsequences. One subsequence is to take $k_n = n^3$ in which case

$$a_{k_n} = [\sqrt[3]{n^3}\,] - \sqrt[3]{n^3} = n - n = 0.$$

Thus this subsequence $(a_{k_n}) = (0)$ has limit 0.

For the other subsequence we use (i) as the hint and try $k_n = n^3 + n^2$. In this case

$$a_{k_n} = [\sqrt[3]{n^3+n^2}\,] - \sqrt[3]{n^3+n^2} \;=\; n - \sqrt[3]{n^3+n^2}.$$

By part (i) this subsequence has limit $-\frac{1}{3}$. Since the two subsequences have different limits the original sequence cannot have a limit.

Question 4(iii) cont. The problem, of course, is that neither $f(x) = 1 - e^{-n}$ nor $g(n) = 2 - e^{-2n}$ tends to zero (or infinity) and so L'Hôpital's rule cannot be applied. One should more simply note that AoL applies to give the correct limit, which is

$$\frac{1-0}{2-0} = \frac{1}{2}.$$