MATH10242 Sequences and Series:
Solutions 6, to exercises for week 7 Tutorials

Question 1: Do the following sequences converge/diverge/tend to infinity or tend to minus infinity?

(*These also appear in the course notes, at the end of Chapter 5.*)

(a) $\big(\cos(n\pi)\sqrt{n}\big)_{n\geq 1}$ (b) $\big(\sin(n\pi)\sqrt{n}\big)_{n\geq 1}$

c) $\left(\dfrac{\sqrt{n^2+2}}{\sqrt{n}}\right)_{n\geq 1}$ (d) $\left(\dfrac{n^3+3^n}{n^2+2^n}\right)_{n\geq 1}$

e) $\left(\dfrac{n^2+2^n}{n^3+3^n}\right)_{n\geq 1}$ (f) $\left(\dfrac{1}{\sqrt{n}-\sqrt{2n}}\right)_{n\geq 1}$

Solutions:

(a) $a_n = \cos(n\pi)\sqrt{n} = (-1)^n\sqrt{n}$. Hopefully it is clear that this ought to diverge, but it also does not tend to ∞ or to $-\infty$. But we should prove it carefully.

None of our rules really applies directly, but what we do know is that $\sqrt{n} \to \infty$ by 5.1.4. In particular $(|a_n|)$ is not bounded, and hence (a_n) is also not bounded. Thus it cannot converge by 2.3.9.

It also does not tend to ∞ simply because it is negative half the time. In other words, with $K = 0$ or $K = 1$ there cannot exist N such that $a_n > K$ for all $n \geq N$. Similarly it cannot tend to $-\infty$, so it just diverges.

(b) $\sin(n\pi)\sqrt{n} = 0$ for all n so the sequence converges (to 0).

(c) We have

$$\frac{\sqrt{n^2+2}}{\sqrt{n}} = \frac{\sqrt{n+2/n}}{\sqrt{1}} \geq \sqrt{n}$$

for $n \geq 1$. By 5.1.4, $\sqrt{n} \to \infty$ and so $\sqrt{n^2+2}/\sqrt{n} \to \infty$ as $n \to \infty$ by the Sandwich Rule 5.1.8(ii).

(d) Note that

$$\frac{n^2+2^n}{n^3+3^n} = \frac{\frac{n^2}{3^n} + (2/3)^n}{\frac{n^3}{3^n}+1} \to \frac{0+0}{0+1} = 0$$

by the AoL and Chapter 4. Hence, by the Reciprocity Theorem 5.1.6,

$$\frac{n^3+3^n}{n^2+2^n} \to \infty$$

as $n \to \infty$.

(e) By the work in (d) we see this has limit 0.

(f) Since $\sqrt{n} - \sqrt{2n} = \sqrt{n}(1 - \sqrt{2}) \to -\infty$, we deduce, by the $-\infty$ version of 5.1.6(i), that

$$\frac{1}{\sqrt{n} - \sqrt{2n}} \to 0$$

as $n \to \infty$.

Question 2: Complete the proof of Theorem 5.1.8, by proving the following result:

Theorem: Suppose that $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ both are sequences that tend to infinity. Prove:

(i) $a_n + b_n \to \infty$ as $n \to \infty$;

(ii) $a_n \cdot b_n \to \infty$ as $n \to \infty$.

(iii) Let $M \in \mathbb{N}$. Assume that $(c_n)_{n\geq 1}$ is a sequence such that $c_n \geq a_n$ for all $n \geq M$. Prove that $c_n \to \infty$ as $n \to \infty$.

Solution: (i) Given $K > 0$ we know, by definition, that there exists N_1 and N_2 such that $a_n > K$ for all $n \geq N_1$ and $b_n > K$ for all $n \geq N_2$. So certainly $a_n + b_n > 2K > K$ for all $n \geq \max\{N_1, N_2\}$.

(ii) Let $K > 0$ and set $K_1 = \max\{K, 1\}$. Then, again, we know that there exists N_1 and N_2 such that $a_n > K_1$ for all $n \geq N_1$ and $b_n > K_1$ for all $n \geq N_2$. So certainly $a_n \cdot b_n > K_1^2 \geq K$ for all $n \geq \max\{N_1, N_2\}$.

(iii) Given $K > 0$ we know that there exists N such that $a_n > K$ for all $n \geq N$. So certainly $c_n \geq a_n > K$ for all $n \geq \max\{N, M\}$.

Question 3: There are many variants on Question 1. Can you think of some? Here is one:

(i) Suppose that $a_n \to \infty$ as $n \to \infty$ and that $(b_n)_{n\geq 1}$ is a sequence of non-zero numbers that converges to $\ell > 0$. Prove that

$$\frac{a_n}{b_n} \to \infty$$

as $n \to \infty$.

(ii) What happens if $\ell = 0$ in part (i)?

Solution: (i) (First proof) As $a_n \to \infty$, certainly $a_n > 0$ for large n. Similarly, $b_n > 0$ for large n (for a detailed proof of this step see the next paragraph). Therefore by the Reciprocal Rule 5.1.6, it is enough to prove that $\lim_{n\to\infty} b_n/a_n = 0$. But, by the Reciprocal Rule, again, $\lim_{n\to\infty} 1/a_n = 0$ and $\lim_{n\to\infty} b_n = \ell$. Thus, by the Algebra of Limits Theorem

$$\lim_{n\to\infty} \frac{1}{a_n} \cdot b_n = 0 \cdot \ell = 0.$$

(Alternative proof) First, there exists N such that $b_n \geq \ell/2$ and $b_n \leq \ell + \ell/2 \leq 2\ell$ for all $n \geq M$. (We have done this trick before, but if you need an extra hint try using the definition of convergence with $\varepsilon = \ell/2$.)

In particular $b_n > 0$ for $n \geq N$. Thus using $b_n \leq 2\ell$ for $n \geq N$ we get that

$$\frac{a_n}{b_n} \geq \frac{a_n}{2\ell}$$

for $n \geq N$. Finally $a_n/2\ell \to \infty$ by Question 2(ii) and hence $a_n/b_n \to \infty$ by Question 2(iii).

(ii) If $\ell = 0$ the sequence (a_n/b_n) either goes to $+\infty$ (for example if $b_n = 1/n$) or to $-\infty$ (for example if $b_n = -1/n$) or it just diverges when the (b_n) oscillate; for example $b_n = (-1)^n/n$.

Question 4: Use the subsequence test to show that:-

(i) the sequence

$$\left(\frac{n}{8} - \left[\frac{n}{8}\right]\right)_{n\geq 1}$$

does not converge;

(ii) the sequence

$$\left(\left[\sin(\frac{n\pi}{4})\right] - \sin\left(\frac{n\pi}{4}\right)\right)_{n\geq 1}$$

does not converge.

Solution: (i) If we take $k_n = 8n$ then $a_{k_n} = a_{8n} = n - [n] = 0$.

On the other hand if $k_n = 8n + 1$

$$a_{k_n} = a_{8n+1} = \frac{8n+1}{8} - \left[\frac{8n+1}{8}\right] = \left(n + \frac{1}{8}\right) - \left[n + \frac{1}{8}\right] = \frac{1}{8}.$$

Since we have two subsequences with distinct limits, Theorem 6.1.3 says that the original sequence (a_n) cannot have a limit.

(ii) This is similar. For $k_n = 8n + 1$

$$a_{k_n} = a_{8n+1} = \left[\sin(\frac{\pi}{4} + 2n\pi)\right] - \sin(\frac{\pi}{4} + 2n\pi) = \left[\frac{1}{\sqrt{2}}\right] - \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}.$$

But if $k_n = 8n$

$$a_{k_n} = a_{8n} = [\sin(2n\pi)] - \sin(2n\pi) = 0.$$

So, again Theorem 6.1.3 says that the original sequence (a_n) cannot have a limit.

Question 5: Assume that $n^{1/\sqrt{n}} \to \ell$ as $n \to \infty$.

Use the subsequence test to show that $\ell = 1$.

[Hint: We do know $\lim_{m\to\infty} m^{1/m}$.]

Solution: Since the hint suggests thinking about $m^{1/m}$, we can try the subsequence (a_{k_n}) for $k_n = n^2$. Then

$$a_{k_n} = a_{n^2} = (n^2)^{1/n} = n^{2/n} = \left(n^{1/n}\right)^2.$$

Now 4.1.5 and the AoL says that this sequence has $\lim_{n\to\infty} a_{k_n} = 1^2 = 1$.

Finally, since we are told that the original sequence (a_n) has a limit ℓ, Theorem 6.1.3 gives $\ell = 1$.

What is harder, and not done here, is to show that the sequence does have a limit.

Question 6*: Prove that $(n!)^{-1/n} \to 0$ as $n \to \infty$. [Hint: Use 4.1.4 with $c = 1/\varepsilon$.]

Solution: Let $\varepsilon > 0$ be given and (by the hint) apply 4.1.4. This implies that

$$\left(\frac{1}{\varepsilon}\right)^n \frac{1}{n!} \to 0$$

as $n \to \infty$. In particular there exists $n \geq 1$ such that

$$\left|\left(\frac{1}{\varepsilon}\right)^n \frac{1}{n!}\right| < 1$$

for all $n \geq N$. Equivalently, $1/n! < \varepsilon^n$ for all $n \geq N$.

Now we take n^{th} roots to get: For all $\varepsilon > 0$ there exists $n \geq 1$ such that $(1/n!)^{\frac{1}{n}} < \varepsilon$ for all $n \geq N$. Which is exactly what we need to prove. □

Solutions to Extra Questions:

Question 7.

(a) Does every bounded increasing sequence converge? **Yes**

(b) Does every increasing sequence of negative terms converge? **Yes** [Note that 0, or 1 or... is an upper bound, so this is a special case of (a).]

(c) Does every decreasing sequence of negative terms converge? **No** [For instance $(-n)_n$.]

(d) Is every bounded sequence convergent? **No** [Standard example: $(-1)^n$.]

(e) Is the limit of an increasing, convergent sequence necessarily equal to the supremum of the set of its terms? **Yes** [For example, one of the main points in the proof of the Monotone Convergence Theorem.]

(f) Let $(a_n)_{n\geq 1}$ be a sequence of nonzero terms. If $\frac{1}{a_n} \to l$ as $n \to \infty$ and $l \neq 0$, does it necessarily follow that the sequence $(a_n)_{n\geq 1}$ converges? **Yes** [Follows from the Algebra of Limits Theorem (vi).]

(g) Let $(a_n)_{n\geq 1}$ be a convergent sequence and let $(b_n)_{n\geq 1}$ be a bounded sequence. Is $(a_n b_n)_{n\geq 1}$ necessarily a convergent sequence? **No** [Take $a_n = 1$, $b_n = (-1)^n$.]

Question 8.

(a) Suppose that the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ converge to a and b respectively. Show that the sequence $(a_n - b_n)_{n\geq 1}$ converges to $a - b$.

Proof: Choose $\varepsilon > 0$. There is N_1 such that $| a_n - a | < \varepsilon/2$ for all $n \geq N_1$. Similarly there is N_2 such that $| b_n - b | < \varepsilon/2$ for all $n \geq N_2$. Set $N = \max\{N_1, N_2\}$. Then, for $n \geq N$, we have

$$| (a_n - b_n) - (a - b) | = | (a_n - a) - (b_n - b) | \leq | a_n - a | + | b_n - b |$$

(by the triangle inequality) $< \varepsilon/2 + \varepsilon/2 = \varepsilon$, as required.

(b) Suppose that the sequence $(a_n)_{n\geq 1}$ converges to a limit ℓ. Suppose also that, for every n, $a_n \leq r$. Prove that $\ell \leq r$.

Proof: Argue by contradiction. If not, then $\ell > r$ so $\ell - r > 0$; set $\varepsilon = \ell - r$. Since $a_n \to \ell$, there is N such that, for all $n \geq N$, we have $|a_n - \ell| < \varepsilon$. So, for $n \geq N$,

$$a_n > \ell - \varepsilon = \ell - (\ell - r) = r,$$

a contradiction, as required [Of course, you don't need to introduce the notation ε, and can just write $r - \ell$ all the way through.]

[An alternative way of finishing off (if you don't notice the faster argument above) from the point where we have $|a_n - \ell| < \varepsilon$:

$$|a_n - \ell| = |\ell - a_n| = |(\ell - r) + (r - a_n)| = (\ell - r) + (r - a_n)$$

(both terms are ≥ 0 since $\ell > r$ and $r \geq a_n$). That is,

$$(\ell - r) + (r - a_n) < \varepsilon = \ell - r,$$

so $r - a_n < 0$, that is $r < a_n$, contradiction, as required.

(c) Suppose that the sequence $(a_n)_{n\geq 1}$ converges to a limit ℓ. Suppose also that there is an integer M such that, for every $n \geq M$, $a_n \leq r$. Is $\ell \leq r$? If so, give a proof; if not, give a counterexample.

Yes. Proof: The proof is almost the same as (b). Just replace the last sentence by: "So, for $n \geq N$,

$$a_n > \ell - \varepsilon = \ell - (\ell - r) = r.$$

So if $n \geq \max\{M, N\}$ then we have both $a_n > r$ and $a_n \leq r$, a contradiction, as required."