MATH10242 Sequences and Series:
Solutions 5, to exercises for week 6 Tutorials

Question 1. Calculate (if they exist) the following limits.
Note: In cases where the limit does not exist, the proof of this fact is a little harder so may be skipped at a first attempt.

(i) $\lim_{n\to\infty}(-\frac{7}{8})^n n^{1000}$; (ii) $\lim_{n\to\infty}\frac{n!}{10^n}$; (iii) $\lim_{n\to\infty}\frac{3^n+n^2}{n^5+3^n}$;

(iv) $\lim_{n\to\infty}\frac{3!}{n^3}$; (v) $\lim_{n\to\infty}\frac{n^n+n!}{n^n+(-1)^n n!}$; (vi) $\lim_{n\to\infty}\frac{n!+n^n}{n!+(-1)^n n^n}$;

Solutions: (i) *Whenever one sees alternating terms (or even negative terms) the following a special case of Theorem 3.1.4(ii) may be useful:* **Theorem** *Suppose that* $(a_n)_{n\in\mathbb{N}}$ *is a sequence for which* $\lim_{n\to\infty}|a_n| = 0$*. Then* $\lim_{n\to\infty} a_n = 0$*.*
Now, for our example, Lemma 4.1.6 gives

$$\lim_{n\to\infty} n^{1000}\cdot\left(\frac{7}{8}\right)^n = 0.$$

Hence, by the above-stated theorem,

$$\lim_{n\to\infty} n^{1000}\cdot\left(\frac{-7}{8}\right)^n = \lim_{n\to\infty}(-1)^N\cdot n^{1000}\cdot\left(\frac{7}{8}\right)^n = 0.$$

(ii) There is no limit. Indeed, by 4.1.4 $\lim_{n\to\infty} 10^n/n! = 0$ which means its reciprocal is unbounded/tends to infinity. This type of observation is generalised in Chapter 5, but here is the detailed proof. Given $\varepsilon = 1/d$ (for any fixed natural number d) there exists N such that

$$0 < \frac{10^n}{n!} < \frac{1}{d}$$

for $n \geq N$. Hence $n!/10^n > d$ for such n and so the sequence is unbounded.
(iii) The term with the highest order of growth is 3^n, so divide top and bottom by it. Then use 4.1.6 and the Algebra of Limits to get

$$\lim_{n\to\infty}\frac{3^n+n^2}{n^5+3^n} = \lim_{n\to\infty}\frac{1+\frac{n^2}{3^n}}{\frac{n^5}{3^n}+1} \to \frac{1+0}{0+1} = 1,$$

as $n\to\infty$.

(iv)

$$\lim_{n\to\infty}\frac{3!}{n^3} = \lim_{n\to\infty}\frac{6}{n^3} = 0.$$

(v) **Remark.** Note that the sequence can only start with

$$a_2 = \frac{2^2+2!}{2^2+2!}$$

since the a_1 term would be $2/0$ which is meaningless. But as we are hoping to understand the limit of a_n as $n\to\infty$, we can ignore the first few terms.

Since n^n is the fastest-growing term, divide by that to get

$$\lim_{n\to\infty} \frac{n^n + n!}{n^n + (-1)^n n!} = \lim_{n\to\infty} \frac{1 + n!/n^n}{1 + (-1)^n n!/n^n}.$$

Now, by 4.1.7(3) $\lim_{n\to\infty}(n!/n^n) = 0$ and hence by the Theorem after part (i) above we get that $\lim_{n\to\infty}((-1)^n n!/n^n) = 0$. Now we can apply the Algebra of Limits to the last display and get

$$\lim_{n\to\infty} \frac{1 + n!/n^n}{1 + (-1)^n n!/n^n} = \frac{1+0}{1+0} = 1.$$

(vi) Here repeating the ideas of part (v) gives

$$\lim_{n\to\infty} \frac{n! + n^n}{n! + (-1)^n n^n} = \lim_{n\to\infty} \frac{n!/n^n + 1}{n!/n^n + (-1)^n}$$

For very large n the first term top and bottom gets very small so the display "looks like" $1/(-1)^n = (-1)^n$ which we know does not converge. So it looks as if it does not converge. We can prove that as follows.

Suppose, for a contradiction, that

$$a_n = \frac{n!/n^n + 1}{n!/n^n + (-1)^n}$$

has a limit, say ℓ. We will argue as we did for $a'_n = (-1)^n$ to get a contradiction. So, to give us a little room take $\varepsilon = 1/4$. Then there exists N such that if $n \geq N$ then $|a_n - \ell| < 1/4$. Open this out as

$$a_n - \frac{1}{4} < \ell < a_n + \frac{1}{4}. \tag{1}$$

Now, if $n \geq N$ is even then $a_n = 1$, so choose $n = 2N$ in (1) when the left hand inequality gives $\ell > 3/4$.

If $n \geq N$ is odd then

$$a_n = \frac{n!/n^n + 1}{n!/n^n - 1}.$$

It is easy to see that if $0 < x < 1$ then

$$\frac{x+1}{x-1} < -1.$$

Thus for odd n, $a_n < -1$. Choose $n = 2N+1$ in (1) when the right hand inequality gives $\ell < -3/4$. We have a contradiction.

Remark: The material of Chapter 6 will give an easier proof of non-convergence.

Question 2. Define $(a_n)_{n\geq 1}$ inductively by $a_1 = 3$, and $a_{n+1} = \dfrac{a_n^2 - 2}{2a_n - 3}$ for $n \geq 1$.

(a) Show for all $n \geq 1$, that $a_n \geq 2$.

(b) Prove that $(a_n)_{n\geq 1}$ is a decreasing sequence.

(c) Deduce that the sequence $(a_n)_{n\geq 1}$ converges and find its limit.

Solution: (a) Certainly $a_1 \geq 2$, so suppose by induction that that $a_n \geq 2$ for some natural number $n \geq 1$. Then

$$\begin{aligned} a_{n+1} = \frac{a_n^2 - 2}{2a_n - 3} \geq 2 \quad &\iff \quad (a_n^2 - 2) \geq 2(2a_n - 3) && \text{since } (2a_n - 3) > 0 \text{ by hypothesis} \\ &\iff \quad a_n^2 - 4a_n + 4 \geq 0 && \text{by collecting terms} \\ &\iff \quad (a_n - 2)^2 \geq 0. \end{aligned}$$

Now this last line is certainly true. Therefore, going backwards through the equivalences, we see that $a_{n+1} \geq 2$. Hence the inductive statement is true for all $n \geq 1$.

Remark: *It is important in such an argument that I have used* $\iff$ *not just* $\Rightarrow$ *between each statement. This is because we want to go back through the implications in the computation.*

$$(a_n - 2)^2 \geq 0 \;\Rightarrow\; (a_n^2 - 2) \geq 2(2a_n - 3) \;\Rightarrow\; a_{n+1} = \frac{a_n^2 - 2}{2a_n - 3} \geq 2.$$

(b) Now

$$\begin{aligned} a_{n+1} = \frac{a_n^2 - 2}{2a_n - 3} \leq a_n \quad &\iff \quad a_n^2 - 2 \leq 2a_n^2 - 3a_n && \text{as } (2a_n - 3) > 0 \\ &\iff \quad 0 \leq a_n^2 - 3a_n + 2 \\ &\iff \quad 0 \leq (a_n - 2)(a_n - 1). \end{aligned}$$

Again this last line is true as $a_n \geq 2$. Thus, we see that $a_{n+1} \leq a_n$ for all $n \geq 1$.

(c) By (a) and (b) (a_n) is decreasing and bounded below. Thus, by the Monotone Convergence Theorem 2.4.3, $\lim_{n\to\infty} a_n$ exists; say $\lim_{n\to\infty} a_n = \ell$. Now, we can use the Algebra of Limits Theorem to see that the sequence $(b_n)_{n\geq 1}$, where

$$b_n = \frac{a_n^2 - 2}{2a_n - 3}$$

also has a limit and that limit is

$$\lim_{n\to\infty} b_n = \frac{\ell^2 - 2}{2\ell - 3}.$$

However, as $(b_n)_{n\geq 1} = (a_{n+1})_{n\geq 1}$, Lemma 4.1.3 says that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n = \ell$. In other words,

$$\ell = \frac{\ell^2 - 2}{2\ell - 3}.$$

Solving we get

$$2\ell^2 - 3\ell = \ell^2 - 2 \qquad \Rightarrow \qquad \ell^2 - 3\ell + 2 = 0 \qquad \Rightarrow \qquad (\ell - 2)(\ell - 1) = 0.$$

Thus either $\ell = 1$ or $\ell = 2$. But, since $a_n \geq 2$ for all n, Lemma 4.2.5 gives that $\ell \geq 2$. Hence $\ell = 2$.

Question 3.

(a) Let $(a_n)_{n\geq 1}$ be a sequence of non-negative real numbers and assume that $a_n \to \ell$ as $n \to \infty$. Set $b_n = \sqrt{a_n}$ for all n.

(i) Assume that $(b_n)_{n\geq 1}$ has a limit. Prove that $b_n \to \sqrt{\ell}$ as $n \to \infty$.

(ii) Assume that the limit of the sequence $(a_n)_{n\geq 1}$ is 0. Prove that $(b_n)_{n\geq 1}$ has a limit and show that $\lim_{n\to\infty} b_n = 0$.

(iii)* Now do as in part (ii) but for any value of ℓ. That is, prove that $(b_n)_n$ does converge and that $\lim_{n\to\infty} b_n = \sqrt{\ell}$.

(b) Hence find $\displaystyle\lim_{n\to\infty} \frac{\sqrt{n+1}+\sqrt{n+2}}{\sqrt{n+3}+\sqrt{2n+4}}$.

Solution: Write $\lim_{n\to\infty} b_n = t$. Then we can use the Algebra of Limits to conclude that $(a_n)_{n\geq 1}$ where $a_n = b_n^2$ has limit t^2. Thus $t^2 = \ell$ and $t = \sqrt{\ell}$ as required.

(ii) Here $\lim_{n\to\infty} a_n = 0$. Pick $\varepsilon > 0$. Then $\varepsilon^2 > 0$ and so there exists $N \in \mathbb{N}$ such that, if $n \geq N$, then $a_n = |a_n - 0| < \varepsilon^2$. Taking square roots gives

$$|b_n| = b_n = \sqrt{a_n} < \sqrt{\varepsilon^2} = \varepsilon \qquad \text{for all } n \geq N.$$

Aside. *The proof of part (iii) will look a bit complicated, so let me begin by doing some natural experiments that will give us a hint towards the proof.*

Suppose that $\lim_{n\to\infty} a_n = \ell \neq 0$. Given $\eta > 0$ pick $N \in \mathbb{N}$ such that $|a_n - \ell| < \eta$ for all $n \geq N$. We want to prove that $b_n = \sqrt{a_n}$ tends to $\sqrt{\ell}$, so lets see what we can say about $|\sqrt{a_n} - \sqrt{\ell}|$. We have seen the trick for dealing with this before:

$$\left|\sqrt{a_n} - \sqrt{\ell}\right| = \left|(\sqrt{a_n} - \sqrt{\ell})\frac{(\sqrt{a_n}+\sqrt{\ell})}{(\sqrt{a_n}+\sqrt{\ell})}\right| = \left|\frac{a_n - \ell}{\sqrt{a_n}+\sqrt{\ell}}\right| < \frac{\eta}{\left|\sqrt{a_n}+\sqrt{\ell}\right|}. \qquad (\dagger)$$

Now, it is clearer. Notice that we do need $\ell \neq 0$ since otherwise we could be dividing by zero in (†). So, for $\ell > 0$ the final term in (†) is $< \eta/\sqrt{\ell}$. So the idea should be to take $\eta = \sqrt{\ell}\,\varepsilon$ and reverse this argument.

So to the proof; we're assuming that $\lim_{n\to\infty} a_n = \ell \neq 0$, and notice that $\ell \geq 0$ by Question 3 and hence $\ell > 0$. In this case given $\varepsilon > 0$ we set $\eta = \varepsilon\sqrt{\ell} > 0$. So we can find $n \geq 1$ such that $|a_n - \ell| < \eta$ for all $n \geq N$. Now (†) applies and gives

$$\left|\sqrt{a_n} - \sqrt{\ell}\right| = \frac{|a_n - \ell|}{\left|\sqrt{a_n}+\sqrt{\ell}\right|} < \frac{\varepsilon\sqrt{\ell}}{\left|\sqrt{a_n}+\sqrt{\ell}\right|} \leq \varepsilon\frac{\sqrt{\ell}}{\sqrt{\ell}} \leq \varepsilon. \qquad \square$$

(b) Now we can divide top and bottom by $\sqrt{n}$ and use part(a) and the Algebra of Limits to get:

$$\frac{\sqrt{n+1}+\sqrt{n+2}}{\sqrt{n+3}+\sqrt{2n+4}} = \frac{\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}}{\sqrt{1+\frac{3}{n}}+\sqrt{2+\frac{4}{n}}} \to \frac{\sqrt{1+0}+\sqrt{1+0}}{\sqrt{1+0}+\sqrt{2+0}} = \frac{2}{1+\sqrt{2}}.$$

Solutions to Extra Questions (more practice; not particularly harder):

Question 4. Determine whether the following sequences converge or not and, in the case of those which do converge, find their limit:

(a) $a_n = \sqrt{\dfrac{2+\sin(n)}{n}}$; (b) $\dfrac{\sin^2(n)}{\sqrt{n}}$;

(c) $n\sin(\pi n)$; (d) $\sqrt[n]{2^{n+1}}$.

Solutions: (a) Using the Algebra of Limits (including the result of Question 3a above), we have:

$$a_n = \sqrt{\frac{2+\sin(n)}{n}} = \sqrt{\frac{2}{n} + \frac{\sin(n)}{n}} \to 0$$

since the sequence $1/n \to 0$ and since $\sin(n)$ is a bounded sequence (so 3.2.2 applies).

(b) Again, this is the product of a null sequence $1/\sqrt{n}$ and a bounded sequence $\sin^2(n)$, so, by 3.2.2, the sequence is null (= has limit 0).

(c) $\sin(\pi n) = 0$ for every integer n, so this is the constant sequence 0 (which has limit 0).

(d)

$$\sqrt[n]{2^{n+1}} = \sqrt[n]{2^n}\,\sqrt[n]{2} = 2 \cdot 2^{1/n}.$$

Using 4.1.1 this converges to 2.

Question 5. Consider the Fibonacci sequence, defined by $a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_n + a_{n+1}$. Consider the sequence defined by $b_n = \dfrac{a_{n+1}}{a_n}$. *Assuming that* the limit of the sequence b_n exists, find it.

Solution:

$$b_n = \frac{a_{n+1}}{a_n} = \frac{a_n + a_{n-1}}{a_n} = 1 + \frac{a_{n-1}}{a_n} = 1 + \frac{1}{b_{n-1}}.$$

Since both b_n and b_{n-1} converge to the same limit, ℓ say, we have (by the Algebra of Limits) $\ell = 1 + 1/\ell$. Multiplying up and rearranging, we get $\ell^2 - \ell - 1 = 0$, so $\ell = (1 \pm \sqrt{5})/2$. Since all the terms a_n are positive, so must be their limit (e.g. by 4.2.5), so we conclude that $\ell = (1+\sqrt{5})/2$ (the Golden Ratio).

Question 6. Define the sequence a_n by $a_1 = 2$, $a_{n+1} = (a_n + 4)/2$. Prove that $a_n < 4$ for every n and that the sequence a_n is monotone increasing. Does this sequence converge? If so, to what limit?

Solution: We use induction to show that $a_n < 4$, the base case ($n = 1$) being given. So assume, inductively, that $a_n < 4$. Then we have

$$a_{n+1} = \frac{1}{2}(a_n + 4) = \frac{a_n}{2} + 2 < 2 + 2$$

since $a_n < 4$.

We don't need induction for the next part now: we have

$$a_{n+1} = \frac{a_n}{2} + 2 > \frac{a_n}{2} + \frac{a_n}{2}$$

(since $2 > a_n/2$), so $a_{n+1} > a_n$, as claimed.

Since the sequence is increasing and bounded above, it converges. Let ℓ be its limit. By the Algebra of Limits we have $\ell = (\ell + 4)/2$, from which we see that $\ell = 4$.