MATH10242 Sequences and Series: Solutions 4, for exercises for week 5 Tutorials

Question 1: By using theorems and examples from the lectures and earlier Examples Sheets, find the limits of the following sequences.

(a) $\left(\dfrac{(2n+1)^2}{n(n^2-n+1)}\right)_{n\geq 1}$ (b) $\left(\dfrac{\cos(n)(2n+1)^2}{n(n^2-n+1)}\right)_{n\geq 1}$

(c) $\left(\dfrac{(2n+1)^3}{n(n^2-n+1)}\right)_{n\geq 1}$ (d) $\left(\dfrac{3^n+5^n}{7^n+9^n}\right)_{n\geq 1}$

(e) $\left(\dfrac{n^3+(\frac{1}{3})^n}{n^3+1}\right)_{n\geq 1}$ (f) $(\sqrt{n^2+n}-\sqrt{n^2-n}\,)_{n\geq 1}$

(g) $\left(\dfrac{n!+1}{(n+1)!}\right)_{n\geq 1}$

[Hint: For part (f), you may also need the following: If $b_n > 0$ for all n and $b_n \to b$ as $n \to \infty$ then $\sqrt{b_n} \to \sqrt{b}$ as $n \to \infty$. Proving this will be one of the exercises on next week's example sheet.]

Solution: In each part of the question, we first try to manipulate the expression (say by dividing top and bottom by some function) in order to get it into a sum of terms for which the limit is obvious. Then we use the Algebra of Limits Theorem 3.2.1 to give the answer. Sometimes you will need to use some results from the notes which we will only be proving this week but which you probably know anyway.

(a) Here we first expand the top and bottom and then divide both by n^3.

$$a_n = \frac{(2n+1)^2}{n(n^2-n+1)} = \frac{4n^2+4n+1}{n^3-n^2+n} = \frac{\frac{4}{n}+\frac{4}{n^2}+\frac{1}{n^3}}{1-\frac{1}{n}+\frac{1}{n^2}}.$$

Now we use the fact that $1/n^r \to 0$ as $n \to \infty$ for $r > 0$ (see Example 3.2.3). Thus by The Algebra of Limits Theorem 3.2.1 parts (ii, iii, v) we get

$$\frac{\frac{4}{n}+\frac{4}{n^2}+\frac{1}{n^3}}{1-\frac{1}{n}+\frac{1}{n^2}} \to \frac{0+0+0}{1-0+0} = \frac{0}{1} = 0 \qquad \text{as } n \to \infty.$$

Remark: In these sorts of questions you do need to be careful about the power of n by which you divide—a higher power than the third would end up with $\frac{0}{0}$ which is useless, while dividing top and bottom by (say) n^2 would leave a term of n in the denominator. Since that does not tend to a limit, you could not then (directly) use 3.2.1.

(b) The sequence is

$$(b_n)_{n\geq 1} = \left(\cos(n)\frac{(2n+1)^2}{n(n^2-n+1)}\right)_{n\geq 1}.$$

Notice that

$$|b_n| \leq \frac{(2n+1)^2}{n(n^2-n+1)},$$

which is the sequence $(a_n)_{n\geq 1}$ from part (a). Thus, by the Sandwich Theorem (the most convenient form is 3.1.4), $\lim_{n\to\infty} b_n = 0$ as well.

(c) Again, we first expand the top and bottom and then divide both by n^3.

$$a_n = \frac{(2n+1)^3}{n(n^2-n+1)} = \frac{8n^3+12n^2+6n+1}{n^3-n^2+n} = \frac{8+12\frac{1}{n}+6\frac{1}{n^2}+\frac{1}{n^3}}{1-\frac{1}{n}+\frac{1}{n^2}}.$$

Applying the same logic as in part (a) we see that

$$\lim_{n\to\infty} a_n = \frac{8+0+0+0}{1-0+0} = 8.$$

(d) Here we divide through top and bottom by 9^n to get

$$a_n = \frac{3^n+5^n}{7^n+9^n} = \frac{(\frac{1}{3})^n+(\frac{5}{9})^n}{(\frac{7}{9})^n+1}.$$

Now by the notes (4.1.2) all but the final entry tends to zero as $n \to \infty$. Thus, by 3.2.1 again

$$\lim_{n\to\infty} a_n = \frac{0+0}{0+1} = \frac{0}{1} = 0.$$

(e) Using 3.2.1 and 4.1.1,

$$\frac{n^3+(\frac{1}{3})^n}{n^3+1} = \frac{1+(\frac{1}{n^3})(\frac{1}{3})^n}{1+\frac{1}{n^3}} \longrightarrow \frac{1+0\cdot 0}{1+0} = 1.$$

(f) We use the same trick for this as on the second sheet:

$$a_n = \sqrt{n^2+n}-\sqrt{n^2-n} = \frac{\left(\sqrt{n^2+n}-\sqrt{n^2-n}\right)\left(\sqrt{n^2+n}+\sqrt{n^2-n}\right)}{\sqrt{n^2+n}+\sqrt{n^2-n}}$$

$$= \frac{(n^2+n)-(n^2-n))}{\sqrt{n^2+n}+\sqrt{n^2-n}} = \frac{2n}{\sqrt{n^2+n}+\sqrt{n^2-n}} = \frac{2}{\sqrt{1+1/n}+\sqrt{1-1/n}}$$

(In the final step, I did the usual thing of dividing top and bottom by n.)
Now, $(1 \pm 1/n) \to 1$ as $n \to \infty$ (use Theorem 3.2.1 again). Using the hint that if $b_n \to 1$ as $n \to \infty$ then $\sqrt{b_n} \to 1$, we have

$$\frac{2}{\sqrt{1+1/n}+\sqrt{1-1/n}} \to \frac{2}{1+1} = 1 \text{ as } n \to \infty.$$

(g) This does not seem to follow directly from our rules, so let's begin by simplifying a bit, by using $(n+1)! = (n!)(n+1)$ and dividing through by $n!$:

$$a_n = \frac{n!+1}{(n+1)!} = \frac{n!+1}{(n+1)(n!)} = \frac{1+\frac{1}{n!}}{(n+1)}. \qquad (\dagger)$$

Now, for example since $0 \le 1/n! \le 1/n$, the Sandwich Rule says $\lim_{n\to\infty} 1/n! = 0$. So now writing

$$a_n = \left(1+\frac{1}{n!}\right)\left(\frac{1}{1+n}\right)$$

and using 3.2.1(iv) gives

$$\lim_{n\to\infty} a_n = (1+0)\cdot 0 = 0.$$

Remark *Note that we cannot directly use Theorem 3.2.1(v) on the displayed equation* (†) *since the sequence* $(n + 1)$ *on the bottom line does not converge.*

Question 2: (a) Prove: **Theorem 3.2.2.** *Let* $(a_n)_{n\geq 1}$ *be a null sequence and let* $(b_n)_{n\geq 1}$ *be a bounded sequence (not necessarily convergent). Then* $(a_n \cdot b_n)_{n\geq 1}$ *is a null sequence.*

Solution: (a) *Proof.* We are given that there exists $B > 0$ such that $|b_n| \leq B$ for all n. Also, for any $\eta > 0$ there exists $N \in \mathbb{N}$ such that $|a_n| < \eta$ for all $n \geq N$.

So, if $\varepsilon > 0$ is given, take $\eta = \varepsilon/B$. Then with N as above we get

$$|a_n b_n| = |a_n| \cdot |b_n| < \eta B = \varepsilon,$$

as required.

(b) True or false: *Let* $(a_n)_{n\geq 1}$ *be a convergent sequence and let* $(b_n)_{n\geq 1}$ *be a bounded sequence (not necessarily convergent). Then* $(a_n \cdot b_n)_{n\geq 1}$ *is a convergent sequence.*

Solution: (b) This is false. For example take $a_n = 1$ for all n (which is certainly convergent) and $b_n = (-1)^n$ (which is certainly bounded). Then $(a_n b_n)_{n\geq 1} = ((-1)^n)_{n\geq 1}$ which is our favourite non-convergent sequence.

Question 3*: Let $(a_n)_{n\geq 1}$ be the sequence defined inductively by $a_1 = 1$, $a_2 = 3$ and, for $n \geq 1$,

$$a_{n+2} = a_{n+1} + a_n.$$

(So $(a_n)_{n\geq 1}$ is like the Fibonacci sequence except that it starts differently.)

(a) Let

$$b = \frac{1+\sqrt{5}}{2} \text{ and } c = \frac{1-\sqrt{5}}{2}.$$

Prove by induction that $\forall n \geq 1, \quad a_n = b^n + c^n$.

(b) Hence find

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Solution: A key to this is to realise that b and c are the roots of the equation $x^2 - x - 1 = 0$.
(a) We compute directly that $a_1 = b + c$ and $a_2 = 3 = b^2 + c^2$.
Assume, inductively, that for some $k \geq 2$ we have that both $\mathcal{P}(k)$ and $\mathcal{P}(k-1)$ are true. Then

$$a_{k+1} \;=\; a_k + a_{k-1} \;=\; b^k + c^k + b^{k-1} + c^{k-1} \;=\; b^{k-1}(b+1) + c^{k-1}(c+1).$$

But b and c are the roots of the equation $x^2 - x - 1 = 0$ and so $b + 1 = b^2$ and $c + 1 = c^2$. Thus

$$a_{k+1} \;=\; b^{k-1}(b+1) + c^{k-1}(c+1) = b^{k+1} + c^{k+1}.$$

Thus, $\mathcal{P}(k+1)$ holds and, by induction $\mathcal{P}(n)$ holds for all n.
(Note that this was a use of "complete induction" where we use, in order to prove $\mathcal{P}(k+1)$, not just the assumption that we have $\mathcal{P}(k)$ but the fact that, in order to get there, we had to prove all the intermediate steps, $\mathcal{P}(m)$ for $m \leq k$.)

(b) First solution: Here we first compute that

$$\frac{c}{b} = \cdots = \frac{6 - 2\sqrt{5}}{-4}$$

and hence that $|c/b| \leq 1/2$. This means that, by the Sandwich Rule, $\lim_{n\to\infty} (c/b)^n \to 0$ as $n \to \infty$.
This gives us the clue about how to manipulate:

$$\frac{a_{n+1}}{a_n} = \frac{b^{n+1} + c^{n+1}}{b^n + c^n} = \frac{b + (\frac{c}{b})^n \cdot c}{1 + (\frac{c}{b})^n}.$$

Hence

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{b + 0 \cdot c}{1 + 0} = b = \frac{1 + \sqrt{5}}{2}.$$

(b) Second solution (longer but, if you don't spot the first solution, this is a fairly natural one, based on investigating the sequence): First notice that

$$\frac{a_{n+1}}{a_n} = \frac{a_n + a_{n-1}}{a_n} = 1 + \frac{a_{n-1}}{a_n}$$

so, *if* the sequence has a limit ℓ say, then that limit should satisfy $\ell = 1 + 1/\ell$, hence be a solution of the equation $x^2 = 1 + x$ that we saw above.

Computing the first few terms of the sequence a_{n+1}/a_n, we see that we can *expect* the limit ℓ to be the larger root, b. Let's see how to *prove* that. We can also see that, after the first few terms, the values seem to stay above 1.5; we'll make use of that observation, as well as the fact that $b = 1 + 1/b$.

Set $r_n = a_n/a_{n-1}$, so we showed in the above paragraph that $r_{n+1} = 1 + 1/r_n$, and consider the difference

$$|r_{n+1} - b| = \left|1 + \frac{1}{r_n} - 1 - \frac{1}{b}\right| = \left|\frac{1}{r_n} - \frac{1}{b}\right| = \frac{|b - r_n|}{br_n}.$$

Since we have $b > 1.5$ and it *seems that* $r_n > 1.5$ for $n \geq 5$, we should have $br_n < 1.25$, hence the difference between r_{n+1} and b decreasing to 0 (by comparison with the sequence $(1/1.25)^n$).

So, to make this into a proper proof we should do the following, noting that $|r_5 - b| < 10^{-1}$. Prove, by induction on $n \geq 5$, that

$$|r_n - b| < 10^{-1} \left(\frac{1}{1.25}\right)^{n-5}$$

and that $r_n > 1.5$. Then we deduce (say, from the definition of convergence) that, since the sequence $(1/1.25)^n$ decreases to 0, the sequence r_n does indeed converge to b. I won't fill in those final details, after all we have another proof above, but leave them as an exercise.

Extra Question* for Week 5: Let x be a real number satisfying $0 < x < 1$. The aim of this question is to prove that $\lim_{n\to\infty} x^n = 0$.

(a) Prove that $(x^n)_{n\geq 1}$ is a convergent sequence and that if ℓ is the limit of this sequence, then $0 \leq \ell < 1$.

We want to show that $\ell = 0$ *so, for the rest of the question, and aiming for a contradiction, suppose that* $\ell > 0$.

(b) Prove that, for any $\varepsilon > 0$ we can find N such that $\ell \leq x^N < \ell + \varepsilon$.

(c) (The punch line) Now we want to mess around with ε (and hence N) to show that for ε small enough we have $(\ell + \varepsilon)x < \ell$ and hence $x^{N+1} < \ell$. This will give the desired contradiction. So, it remains to prove:

$(*)$ Let $\varepsilon = \ell \cdot (\frac{1}{x} - 1)$ Prove that $\varepsilon > 0$ and that, if y satisfies $0 < y < \ell + \varepsilon$, then $0 < yx < \ell$. Then use this to prove (c).

Solution: (a) Clearly each $x^n > 0$. Thus, as $0 < x < 1$, it follows (by induction on n) that $0 < x^n \cdot x < x^n \cdot 1$ for all $n \geq 0$. That is, $(x^n)_{n\geq 1}$ is a strictly decreasing sequence bounded below by zero. Hence, by the Monotone Convergence Theorem (the version we saw on last week's Exercise Sheet), that sequence has a limit, say ℓ. Since $0 < x^n$ for each n it follows that $0 \leq \ell$. (Can you fill in the details, using the definition of convergence, of this last statement? [Suppose, for a contradiction, that $\ell < 0$, ...])

(b) Given $\ell > 0$, there is, by the definition of convergence, some N such that $\ell - \varepsilon < x^n < \ell + \varepsilon$ for all $n \geq N$. Since $(x^n)_{n\geq 1}$ is decreasing, $\ell \leq x^n$ for each n (by the proof of the Monotone Convergence Theorem, ℓ is the infimum of the set of x^n). Thus we get $\ell \leq x^n < \ell + \varepsilon$ for all $n \geq N$, in particular for $n = N$ itself.

(c) Set $\varepsilon = \ell \cdot (1/x - 1)$; note that, since $x < 1$, we have $1/x > 1$ and so $\varepsilon > 0$. Then $\varepsilon + \ell = \ell/x$.

Then for any y with $0 < y < \ell + \varepsilon$ we have

$$0 < yx < (\ell + \varepsilon)x = (\ell/x)x = \ell.$$

Then take N as in part (b). Set $y = x^N$ and apply $(*)$ to get $x^{N+1} = yx < \ell$. But this contradicts the fact that $\ell \leq x^n$ for all n, as required.

Remark For an alternative and easier proof see 4.1.2 in the course notes.