

MATH10242 Sequences and Series:
Solutions 10, for exercises for week 11 Tutorials

Question 1: Find the radius of convergence R of the following power series.

In parts (i) and (ii), what is the *interval of convergence* of the given power series?

$$(i) \sum_{n \geq 1} \frac{x^n}{8^n} \quad (ii) \sum_{n \geq 1} \frac{(-x)^n}{4n+1}, \quad (iii) \sum_{n \geq 1} \frac{(2n)!}{(n!)^2} x^n, \quad (iv) \sum_{n \geq 1} \frac{n^n}{n!} x^n,$$

$$(v) \sum_{n \geq 1} n! \cdot x^n \quad (vi) \sum_{n \geq 1} \frac{\sqrt{(2n)!}}{n!} x^n.$$

[You will need to use the formula for $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$ from a previous Exercise Sheet.]

Solutions: (i) $\sum_{n \geq 1} c_n = \sum_{n \geq 1} x^n/8^n$. Here the (Modified) Ratio Test gives

$$\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{x^{n+1}}{x^n} \frac{8^n}{8^{n+1}} \right| = \frac{|x|}{8}.$$

Thus, (the limit as $n \rightarrow \infty$ of) $|c_{n+1}/c_n| < 1 \iff |x| < 8$ (in which case it converges absolutely) and (the limit as $n \rightarrow \infty$ of) $|c_{n+1}/c_n| > 1 \iff |x| > 8$ (in which case it diverges).

Thus, the radius of convergence is $R = 8$.

When $x = 8$, respectively $x = -8$ the series becomes $\sum_{n \geq 1} 1$, respectively $\sum_{n \geq 1} (-1)^n$ and clearly neither of these converges (use the n^{th} term test). Thus the interval of convergence is $(-8, 8)$.

(ii) $\sum_{n \geq 1} c_n = \sum_{n \geq 1} (-x)^n/(4n+1)$ Here the (Modified) Ratio Test gives

$$\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{x^{n+1}}{x^n} \frac{4n+1}{4(n+1)+1} \right| = |x| \frac{4n+1}{4n+5} = |x| \frac{\left(1 + \frac{1}{4n}\right)}{\left(1 + \frac{5}{4n}\right)} \rightarrow |x| \text{ as } n \rightarrow \infty.$$

Thus, $\lim_{n \rightarrow \infty} |c_{n+1}/c_n| < 1 \iff |x| < 1$ (in which case it converges absolutely) and $\lim_{n \rightarrow \infty} |c_{n+1}/c_n| > 1 \iff |x| > 1$ (in which case it diverges).

Thus, the radius of convergence is $R = 1$.

When $x = 1$ the series becomes

$$\sum_{n \geq 1} \frac{(-1)^n}{4n+1},$$

which converges by the Alternating Series Test.

When $x = -1$ the series becomes

$$\sum_{n \geq 1} \frac{(+1)}{4n+1},$$

which diverges, for example by comparison with $\sum_{n \geq 1} 1/n$.

Thus the interval of convergence is $(-1, 1]$.

(iii) Here $\sum_{n \geq 1} c_n = \sum_{n \geq 1} (2n)!x^n/(n!)^2$. Thus

$$\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{x^{n+1} (2n+2)!}{x^n (2n)!} \cdot \frac{(n!)^2}{((n+1)!)^2} \right| = |x| \frac{(2n+1)(2n+2)}{(n+1)^2} = |x| \frac{4 + \frac{6}{n} + \frac{2}{n^2}}{1 + \frac{2}{n} + \frac{1}{n^2}} \rightarrow 4|x|,$$

as $n \rightarrow \infty$.

Thus, $\lim_{n \rightarrow \infty} |c_{n+1}/c_n| < 1 \iff |x| < 1/4$ (in which case it converges absolutely) and $\lim_{n \rightarrow \infty} |c_{n+1}/c_n| > 1 \iff |x| > 1/4$ (in which case it diverges).

Thus, the radius of convergence is $R = 1/4$.

(iv) Here

$$\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{x^{n+1} (n+1)^{n+1} n!}{x^n n^n (n+1)!} \right| = |x| \frac{(n+1)(n+1)^n}{n^n (n+1)} = |x| \frac{(n+1)^n}{n^n} = |x| \left(1 + \frac{1}{n}\right)^n \rightarrow |x| \cdot e,$$

as $n \rightarrow \infty$.

Thus, by the Ratio Test, again, the radius of convergence is $R = e^{-1}$.

(v) Here

$$\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{x^{n+1}}{x^n} \frac{(n+1)!}{n!} \right| = (n+1)|x| \rightarrow \infty, \quad \text{as } n \rightarrow \infty.$$

So, by the Ratio Test there is no value of x for which the series converges (except of course for $x = 0$) and so the Radius of Convergence is $R = 0$.

(vi) $\sum_{n \geq 1} c_n = \sum_{n \geq 1} \sqrt{(2n)!} x^n / n!$. Thus

$$\begin{aligned} \left| \frac{c_{n+1}}{c_n} \right| &= \left| \frac{\sqrt{(2n+2)!}}{(n+1)!} x^{n+1} \cdot \frac{n!}{x^n \sqrt{(2n)!}} \right| \\ &= \left| \frac{\sqrt{(2n+2)(2n+1)(2n)!}}{(n+1)!} x^{n+1} \cdot \frac{n!}{x^n \sqrt{(2n)!}} \right| \\ &= \frac{\sqrt{(2n+2)(2n+1)}}{(n+1)} \cdot |x| \\ &= \frac{\sqrt{(2+2/n)(2+1/n)}}{(1+1/n)} \cdot |x| \rightarrow 2|x| \end{aligned}$$

as $n \rightarrow \infty$. Thus the radius of convergence is $R = 1/2$.

Question 2: Let $r > 0$. Using Question 1(i) as a guide, find a series $\sum_{n=1}^{\infty} a_n x^n$ with radius of convergence r .

Solution: Replacing 8 by r in Question 1(i) we should guess that the series we want is

$$\sum_{n \geq 1} \frac{x^n}{r^n}.$$

And, this does indeed work since using the Ratio Test again we see that:

$$\left| \frac{c_{n+1}}{c_n} \right| = \left| \frac{x^{n+1}}{x^n} \frac{r^n}{r^{n+1}} \right| = \frac{|x|}{r}.$$

Thus, $\lim_{n \rightarrow \infty} |c_{n+1}/c_n| < 1 \iff |x| < r$ (in which case it converges absolutely) and $\lim_{n \rightarrow \infty} |c_{n+1}/c_n| > 1 \iff |x| > r$ (in which case it diverges).

Thus, the radius of convergence is $R = r$.

Question 3: Let $\sum_{n \geq 1} a_n$ be a series. We define two new series $\sum_{n \geq 1} a_n^+$, consisting of all the positive terms of the original series and $\sum_{n \geq 1} a_n^-$, consisting of all the negative terms. To be specific, set

$$a_n^+ = \frac{a_n + |a_n|}{2} \quad \text{and} \quad a_n^- = \frac{a_n - |a_n|}{2},$$

and notice that if $a_n > 0$ then $a_n^+ = a_n$ and $a_n^- = 0$. Conversely, if $a_n < 0$ then $a_n^- = a_n$ and $a_n^+ = 0$.

(a) Prove that, if $\sum_{n \geq 1} a_n$ is absolutely convergent, then both $\sum_{n \geq 1} a_n^+$ and $\sum_{n \geq 1} a_n^-$ are convergent. Moreover, prove that

$$\sum_{n \geq 1} a_n = \sum_{n \geq 1} a_n^+ + \sum_{n \geq 1} a_n^-.$$

(b*) Prove that, if $\sum_{n \geq 1} a_n$ is only conditionally convergent, then both $\sum_{n \geq 1} a_n^+$ and $\sum_{n \geq 1} a_n^-$ are divergent.

Solution: We always use the description $a_n^+ = (a_n + |a_n|)/2$ and $a_n^- = (a_n - |a_n|)/2$.

(a) (This is really the same as the proof of Theorem 10.2.2, but let's prove it directly.)

Suppose first that $\sum_{n \geq 1} a_n$ is absolutely convergent; thus $\sum_{n \geq 1} |a_n|$ converges. Hence by the Algebra of Infinite Sums (Theorem 8.1.5),

$$\frac{1}{2} \sum_{n \geq 1} (a_n + |a_n|)$$

also converges. By definition this is exactly $\sum_{n \geq 1} a_n^+$. The argument for $\sum_{n \geq 1} a_n^-$ is similar.

Finally, as $a_n = a_n^+ + a_n^-$ for all n , Theorem 8.1.5 then implies that

$$\sum_{n \geq 1} a_n = \sum_{n \geq 1} a_n^+ + \sum_{n \geq 1} a_n^-.$$

(b) Now suppose that $\sum_{n \geq 1} a_n$ is only conditionally convergent; thus $\sum |a_n|$ diverges. In this case, $\sum_{n \geq 1} \frac{1}{2} a_n$ is still convergent while $\sum \frac{1}{2} |a_n|$ diverges. Thus by Question 3(a) on the Week 9 Exercises Sheet,

$$\sum_{n \geq 1} \left(\frac{1}{2} a_n + \frac{1}{2} |a_n| \right)$$

also diverges. By definition this is exactly $\sum_{n \geq 1} a_n^+$. The argument for $\sum_{n \geq 1} a_n^-$ is similar.