MATH10242 Sequences and Series: Exercises 6, for Week 7 Tutorials

You should attempt (at the very least!) Question 1 from this sheet.

Question 1: Do the following sequences converge/diverge/tend to infinity or tend to minus infinity?

(*These also appear in the course notes, at the end of Chapter 5.*)

(a) $\left(\cos(n\pi)\sqrt{n}\right)_{n\geq 1}$ (b) $\left(\sin(n\pi)\sqrt{n}\right)_{n\geq 1}$

c) $\left(\dfrac{\sqrt{n^2+2}}{\sqrt{n}}\right)_{n\geq 1}$ (d) $\left(\dfrac{n^3+3^n}{n^2+2^n}\right)_{n\geq 1}$

e) $\left(\dfrac{n^2+2^n}{n^3+3^n}\right)_{n\geq 1}$ (f) $\left(\dfrac{1}{\sqrt{n}-\sqrt{2n}}\right)_{n\geq 1}$

Question 2: Complete the proof of Theorem 5.1.8, by proving the following result:

Theorem: Suppose that $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ both are sequences that tend to infinity. Prove:

(i) $a_n + b_n \to \infty$ as $n \to \infty$;

(ii) $a_n \cdot b_n \to \infty$ as $n \to \infty$.

(iii) Let $M \in \mathbb{N}$. Assume that $(c_n)_{n\geq 1}$ is a sequence such that $c_n \geq a_n$ for all $n \geq M$. Prove that $c_n \to \infty$ as $n \to \infty$.

Question 3: There are many variants on Question 2. Can you think of some? Here is one:

(i) Suppose that $a_n \to \infty$ as $n \to \infty$ and that $(b_n)_{n\geq 1}$ is a sequence of non-zero numbers that converges to $\ell > 0$. Prove that $a_n/b_n \to \infty$ as $n \to \infty$.

(ii) What happens if $\ell = 0$ in part (i)?

Question 4: Use the subsequence test to show that:-

(i) the sequence $\left(\dfrac{n}{8} - \left[\dfrac{n}{8}\right]\right)_{n\geq 1}$ does not converge;

(ii) the sequence $\left(\left[\sin(\frac{n\pi}{4})\right] - \sin(\frac{n\pi}{4})\right)_{n\geq 1}$ does not converge.

Question 5: Assume that $n^{1/\sqrt{n}} \to \ell$ as $n \to \infty$.

Use the subsequence test to show that $\ell = 1$. [Hint: We do know $\lim\limits_{m\to\infty} m^{1/m}$.]

Question 6*: Prove that $(n!)^{-1/n} \to 0$ as $n \to \infty$. [Hint: Use 4.1.4 with $c = 1/\varepsilon$.]

Extra Questions for Week 7:

Question 7.

(a) Does every bounded increasing sequence converge?

(b) Does every increasing sequence of negative terms converge?

(c) Does every decreasing sequence of negative terms converge?

(d) Is every bounded sequence convergent?

(e) Is the limit of an increasing, convergent sequence necessarily equal to the supremum of the set of its terms?

(f) Let $(a_n)_{n\geq 1}$ be a sequence of nonzero terms. If $1/a_n \to l$ as $n \to \infty$ and $l \neq 0$, does it necessarily follow that the sequence $(a_n)_{n\geq 1}$ converges?

(g) Let $(a_n)_{n\geq 1}$ be a convergent sequence and let $(b_n)_{n\geq 1}$ be a bounded sequence. Is $(a_n b_n)_{n\geq 1}$ necessarily a convergent sequence?

Question 8.

(a) Suppose that the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ converge to a and b respectively. Show that the sequence $(a_n - b_n)_{n\geq 1}$ converges to $a - b$.

(b) Suppose that the sequence $(a_n)_{n\geq 1}$ converges to a limit ℓ. Suppose also that, for every n, $a_n \leq r$. Prove that $\ell \leq r$.

(c) Suppose that the sequence $(a_n)_{n\geq 1}$ converges to a limit ℓ. Suppose also that there is an integer M such that, for every $n \geq M$, $a_n \leq r$. Is $\ell \leq r$? If so, give a proof; if not, give a counterexample.