MATH10242 Sequences and Series: Exercises 5, for Week 6 Tutorials

The tutorial should not be your first sight of the questions! - try to completely answer at least a selection of them before the tutorial. You must ensure that you understand how to do the non-starred questions, in particular the more computational Questions 0 (the starter question), 1 and 2.

Question 0: *This is supposed to be an easier set of questions to get you started, and the solutions are given on the second page of this question sheet.* Calculate (if they exist) the following limits. In cases where the limit does not exist, the proof of this fact is a little harder, so skip that if you don't see how to do it.

(i) $\lim_{n\to\infty}(-\frac{1}{8})^n$; (ii) $\lim_{n\to\infty}\frac{n!}{2^n}$; (iii) $\lim_{n\to\infty}\frac{2^n+n}{n^2+2^n}$; (iv) $\lim_{n\to\infty}\frac{2+n}{n^2+2^n}$;

Question 1. Calculate (if they exist) the following limits.
Note: In cases where the limit does not exist, the proof of this fact is a little harder so may be skipped at a first attempt.

(i) $\lim_{n\to\infty}(-\frac{7}{8})^n n^{1000}$; (ii) $\lim_{n\to\infty}\frac{n!}{10^n}$; (iii) $\lim_{n\to\infty}\frac{3^n+n^2}{n^5+3^n}$;

(iv) $\lim_{n\to\infty}\frac{3!}{n^3}$; (v) $\lim_{n\to\infty}\frac{n^n+n!}{n^n+(-1)^n n!}$; (vi) $\lim_{n\to\infty}\frac{n!+n^n}{n!+(-1)^n n^n}$;

Question 2. Define $(a_n)_{n\geq 1}$ inductively by $a_1 = 3$, and

$$a_{n+1} = \frac{a_n^2 - 2}{2a_n - 3}$$

for $n \geq 1$.

(a) Show for all $n \geq 1$, that $a_n \geq 2$.

(b) Prove that $(a_n)_{n\geq 1}$ is a decreasing sequence.

(c) Deduce that the sequence $(a_n)_{n\geq 1}$ converges and find its limit.

Question 3.

(a) Let $(a_n)_{n\geq 1}$ be a sequence of non-negative real numbers and assume that $a_n \to \ell$ as $n \to \infty$. Set $b_n = \sqrt{a_n}$ for all n.

(i) Assume that $(b_n)_{n\geq 1}$ has a limit. Prove that $b_n \to \sqrt{\ell}$ as $n \to \infty$.

(ii) Assume that the limit of the sequence $(a_n)_{n\geq 1}$ is 0. Prove that $(b_n)_{n\geq 1}$ has a limit and show that $\lim_{n\to\infty} b_n = 0$.

(iii)* Now do as in part (ii) but for any value of ℓ. That is, prove that $(b_n)_n$ does converge and that $\lim_{n\to\infty} b_n = \sqrt{\ell}$.

(b) Hence find $\lim_{n\to\infty}\frac{\sqrt{n+1}+\sqrt{n+2}}{\sqrt{n+3}+\sqrt{2n+4}}$.

Question 4. Determine whether the following sequences converge or not and, in the case of those which do converge, find their limit:

(a) $a_n = \sqrt{\dfrac{2+\sin(n)}{n}}$; (b) $\dfrac{\sin^2(n)}{\sqrt{n}}$;

(c) $n\sin(\pi n)$; (d) $\sqrt[n]{2^{n+1}}$.

Question 5. Consider the Fibonacci sequence, defined by $a_1 = 1$, $a_2 = 1$, $a_{n+2} = a_n + a_{n+1}$. Consider the sequence defined by $b_n = \dfrac{a_{n+1}}{a_n}$. *Assuming that* the limit of the sequence b_n exists, find it.

Question 6. Define the sequence a_n by $a_1 = 2$, $a_{n+1} = \frac{1}{2}(a_n + 4)$. Prove that $a_n < 4$ for every n and that the sequence a_n is monotone increasing. Does this sequence converge? If so, to what limit?

Solutions to Question 0.

(i) Here, notice that $\lim_{n\to\infty} 1/8^n = 0$ by 4.1.2 and hence $\lim_{n\to\infty} (-1/8)^n = 0$ by 3.1.4.

(ii) $\lim_{n\to\infty} n!/2^n$ does not converge (or, if you prefer tends to infinity) since $n!$ has higher order of growth than 2^n (use Lemma 4.1.4 in the notes to deduce that).

(iii) Here 2^n is the fastest-growing term so divide top and bottom by it:

$$\lim_{n\to\infty} \frac{2^n + n}{n^2 + 2^n} = \lim_{n\to\infty} \frac{1 + 2^{-n}n}{2^{-n}n^2 + 1} \to \frac{1+0}{0+1} = 1,$$

where I have used 4.1.6 and the AoL (Algebra of Limits Theorem.)

(iv) Again 2^n is the term with the highest order of growth and so

$$\lim_{n\to\infty} \frac{2 + n}{n^2 + 2^n} = \lim_{n\to\infty} \frac{2^{1-n} + 2^{-n}n}{2^{-n}n^2 + 1} \to \frac{0+0}{0+1} = 0,$$

where we have also used 4.1.4 and the Algebra of Limits Theorem.