MATH10242 Sequences and Series: Exercises 4, for Week 5 Tutorials

As always, you must ensure that you understand how to do the non-starred questions. Question 0 is the starter question, with solutions on the next page. Questions 1 and 2 both are key question types.

Question 0: By using theorems and examples from the lectures and earlier Examples Sheets, find the limits of the following sequences. For parts (c) and (d) you can quote Lemma 4.1.2 (which we will cover soon).

(a) $\left(\dfrac{n^3+1}{n(n^2-1)}\right)_{n\geq 1}$ (b) $\left(\dfrac{n^3+1}{n^2(n^2-1)}\right)_{n\geq 1}$

(c) $\left(\dfrac{3^n}{4^n}\right)_{n\geq 1}$ (d) $\left(\dfrac{3^n+1}{3^n+4}\right)_{n\geq 1}$

Question 1: By using theorems and examples from the lectures and earlier Examples Sheets, find the limits of the following sequences.

(a) $\left(\dfrac{(2n+1)^2}{n(n^2-n+1)}\right)_{n\geq 1}$ (b) $\left(\dfrac{\cos(n)(2n+1)^2}{n(n^2-n+1)}\right)_{n\geq 1}$

(c) $\left(\dfrac{(2n+1)^3}{n(n^2-n+1)}\right)_{n\geq 1}$ (d) $\left(\dfrac{3^n+5^n}{7^n+9^n}\right)_{n\geq 1}$

(e) $\left(\dfrac{n^3+(\frac{1}{3})^n}{n^3+1}\right)_{n\geq 1}$ (f) $(\sqrt{n^2+n}-\sqrt{n^2-n}\,)_{n\geq 1}$

(g) $\left(\dfrac{n!+1}{(n+1)!}\right)_{n\geq 1}$

[Hint: For part (f), you may also need the following: If $b_n > 0$ for all n and $b_n \to b$ as $n \to \infty$ then $\sqrt{b_n} \to \sqrt{b}$ as $n \to \infty$. Proving this will be one of the exercises on next week's example sheet.]

Question 2: (a) Prove: **Theorem 3.2.2.** *Let $(a_n)_{n\geq 1}$ be a null sequence and let $(b_n)_{n\geq 1}$ be a bounded sequence (not necessarily convergent). Then $(a_n \cdot b_n)_{n\geq 1}$ is a null sequence.*

(b) True or false: *Let $(a_n)_{n\geq 1}$ be a convergent sequence and let $(b_n)_{n\geq 1}$ be a bounded sequence (not necessarily convergent). Then $(a_n \cdot b_n)_{n\geq 1}$ is a convergent sequence.*

Note that with a "true or false" question you must either prove it is true or find an example showing that it is false.

Question 3*: Let $(a_n)_{n\geq 1}$ be the sequence defined inductively by $a_1 = 1$, $a_2 = 3$ and, for $n \geq 1$,

$$a_{n+2} = a_{n+1} + a_n.$$

(So $(a_n)_{n\geq 1}$ is like the Fibonacci sequence except that it starts differently.)

(a) Let

$$b = \frac{1+\sqrt{5}}{2} \text{ and } c = \frac{1-\sqrt{5}}{2}.$$

Prove by induction that $\forall n \geq 1,\ \ a_n = b^n + c^n$.

(b) Hence find

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Extra Question* for Week 5: Let x be a real number satisfying $0 < x < 1$. The aim of this question is to prove that $\lim_{n\to\infty} x^n = 0$.

(a) Prove that $(x^n)_{n\geq 1}$ is a convergent sequence and that if ℓ is the limit of this sequence, then $0 \leq \ell < 1$.

We want to show that $\ell = 0$ *so, for the rest of the question, and aiming for a contradiction, suppose that* $\ell > 0$.

(b) Prove that, for any $\varepsilon > 0$ we can find N such that $\ell \leq x^N < \ell + \varepsilon$.

(c) (The punch line) Now we want to mess around with ε (and hence N) to show that for ε small enough we have $(\ell + \varepsilon)x < \ell$ and hence $x^{N+1} < \ell$. This will give the desired contradiction. So, it remains to prove:

($*$) Let $\varepsilon = \ell \cdot (\frac{1}{x} - 1)$ Prove that $\varepsilon > 0$ and that, if y satisfies $0 < y < \ell + \varepsilon$, then $0 < yx < \ell$. Then use this to prove (c).

Solutions to Question 0.

(a) Just as we did in Example 3.2.4, we divide top and bottom by the leading power of x (that is, by n^3):

$$a_n = \frac{n^3+1}{n(n^2-1)} = \frac{n^3+1}{n^3-n} = \frac{1+\frac{1}{n^3}}{1-\frac{1}{n^2}}.$$

Now, by Example 3.2.3, if $r=2$ or $r=3$ then $1/n^r \to 0$ as $n\to\infty$. So, we can use the Algebra of Limits Theorem to get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1+\frac{1}{n^3}}{1-\frac{1}{n^2}} = \frac{1+0}{1-0} = 1.$$

(b) Using the same ideas as in (a) we see that

$$\frac{n^3+1}{n^2(n^2-1)} = \frac{n^3+1}{n^4-n^2} = \frac{\frac{1}{n}+\frac{1}{n^4}}{1-\frac{1}{n^2}} \to \frac{0+0}{1+0} = 0 \quad as \quad n\to\infty.$$

Note that in this example, if you divided top and bottom by n^3 you would get an "n" in the denominator, which would require extra work.

(c) Here one needs to use Lemma 4.1.2 which says that for $0<c<1$ we have $c^n \to 0$ as $n\to\infty$. So notice that

$$\frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n \to 0 \quad as \quad n\to\infty.$$

(d) In this case we need to divide top and bottom by 3^n in order to use Lemma 4.1.2. But then combining that result with the Algebra of Limits Theorem, it all does work:

$$\frac{3^n+1}{3^n+4} = \frac{1+(\frac{1}{3})^n}{1+4(\frac{1}{3})^n} \to \frac{1+0}{1+4\cdot 0} = 1.$$