MATH10242 Sequences and Series: Exercises 3, for Week 4 Tutorials

Question 0 is an easier question on which to get started. The solution to it is given on the next page — but don't look at the answers until you have worked seriously at the question! (a general point!) Question 1 is the most important on this sheet.

Question 0: Which of the following sequences converge (and to what number)? You should justify your answers. In some of the questions you may be able to use the theory we have been developing, but for others you may have to compute them explicitly as we did last week.

(a)
$$\left(1 - \frac{1}{2n^3}\right)_{n\geq 1};$$

(b)
$$\left(1 + n^3\right)_{n\geq 1};$$

(c)
$$\left(\sqrt{n+1} - \sqrt{n}\right)_{n\geq 1};$$

[See last week's solutions.]

(d)
$$\left(\frac{n^3}{5^n}\right)_{n\geq 1}.$$

[See examples in the notes.]

Question 1: Which of the following sequences converge (and to what number)? Justify your answers.

(a)
$$\left(1 - \frac{3n^3 + n^2}{2n^3}\right)_{n\geq 1};$$

(b)
$$\left(1 - \frac{3n^2 + n^3}{2n^2}\right)_{n\geq 1}$$

(c)
$$\left(\sqrt{n^2+1} - n\right)_{n\geq 1};$$

[Hint: Use the ideas we used for $\sqrt{n+2} - \sqrt{n}$]

(d)
$$\left(\sqrt{2n} - \sqrt{n}\right)_{n\geq 1};$$

(e)
$$\left(3^{-n}\right)_{n\in\mathbb{N}};$$

(f)
$$\left(\frac{n^3}{3^n + 4^n}\right)_{n\geq 1}.$$

Question 2: Let $(a_n)_{n\geq 1}$ be a bounded, decreasing sequence. Prove that $(a_n)_{n\geq 1}$ is convergent.

Question 3: a. Define the sequence $(a_n)_{n\geq 1}$ inductively by $a_1 = 1$ and $a_{n+1} = a_n/3 + 1$.
i. Prove that $a_n \leq 3/2$ for all $n \geq 1$,
ii. Prove that $(a_n)_{n\geq 1}$ is an increasing sequence.
iii. What Theorem implies the sequence converges? Show that the limit is 3/2.

b. What happens if the starting value is $a_1 = 4$?

Question 4 What is

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\cdots}}}}?$$

Hint Consider it the limit (if it exists) of

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}, ...$$

i.e 1.414..., 1.8477..., 1.961..., 1.990..., 1.997..., 1.999...

i. The evidence here is that the sequence is increasing. Prove it is.
ii. Is it bounded above? Prove that it is.
iii. What Theorem implies the sequence converges? Show that the limit is 2.

Extra Question for Week 4: Suppose that $p(x)$ and $q(x)$ are polynomials with real coefficients, and $q(x) \neq 0$. What is the limit of the sequence $a_n = p(n)/q(n)$ as $n \to \infty$?

Solution to Question 0.

(a) Likely you guessed the limit to be $\ell = 1$. So, let's prove it. First,

$$|a_n - 1| = \left|1 - \frac{1}{2n^3} - 1\right| = \frac{1}{2n^3} < \frac{1}{n}.$$

Now, given $\varepsilon > 0$ if we take $N = [1/\varepsilon] + 1$, then for $n \geq N$ we get $1/n < \varepsilon$ and hence $|a_n - 1| < \varepsilon$ and we are done.

You could also use the notes more directly: Check that $(b_n)_{n\geq 1} = (1 - 1/n)_{n\geq 1}$ does indeed converge to 1 and then use the Sandwich Theorem.

(b) Since $1 + n^3 \geq n$ for all natural numbers n, the sequence is not bounded (see Example 2.4.8). Thus the sequence $(1 + n^3)_{n\geq 1}$ does not converge, by Theorem 2.3.9.

(c)

$$\begin{aligned}
\sqrt{n+1} - \sqrt{n} &= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{(\sqrt{n+1} + \sqrt{n})} \\
&= \frac{\left((\sqrt{n+1})^2 - (\sqrt{n})^2\right)}{\sqrt{n+1} + \sqrt{n}} = \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\
&\leq \frac{1}{2\sqrt{n}} \\
&\leq \frac{1}{\sqrt{n}}.
\end{aligned}$$

So, just as in the (very) similar example from last week, we need $1/\sqrt{n} < \varepsilon$ or $n > \varepsilon^{-2}$. So, take $N = 1 + [\varepsilon^{-2}]$.

Then, again, tracing back through our computations, we see that if $n \geq N$ then $1/\sqrt{n} < \varepsilon$ and so $\sqrt{n+1} - \sqrt{n} < \varepsilon$. In other words, the limit is 0.

(d) Since $5^n > 4^n$, clearly

$$0 < \frac{n^3}{5^n} < \frac{n^3}{4^n} = \frac{n^3}{2^n \cdot 2^n}.$$

Now, by the notes $2^n > n^2$, for $n \geq 5$ and hence

$$0 < \frac{n^3}{5^n} < \frac{n^3}{n^4} = \frac{1}{n} \qquad \text{for } n \geq 5.$$

By the notes, again, (see 3.1.5) we know that $(1/n)_{n\geq 1}$ is null and so by 3.1.4(ii) our given sequence $(a_n) = (n^3/5^n)_{n\geq 1}$ is null.

Alternatively, you could prove by induction that, for all $n \geq 4$, we have $5^n \geq n^4$.