MATH10242 Sequences and Series: Exercises 2, for Week 3 Tutorials

Questions 2 and 3 are important, so be sure to spend enough time on them. As usual, an asterisk indicates a significantly harder problem.

Question 1: Here you should justify a couple of formulas that we will often use. Prove:
(a) $\forall x, y, \quad |x - y| \geq |\, |x| - |y| \,|$;
(b) $\forall x, \ell$ and $\forall \varepsilon > 0$, $|x - \ell| < \varepsilon \iff \ell - \varepsilon < x < \ell + \varepsilon$.

Question 2: Let $\varepsilon > 0$ be given. For each of the following sequences (a_n), find a natural number N such that $\forall n \geq N$, one has $|a_n| < \varepsilon$ (thereby showing that $a_n \to 0$ as $n \to \infty$).

(a)
$$a_n = \frac{1}{n^2}.$$

(b)
$$a_n = \frac{n + \sqrt{n}}{n^2 + 1}.$$

(c)
$$a_n = \frac{\cos n}{n}.$$

(d)
$$a_n = \sqrt{n+2} - \sqrt{n}$$

(e)*
$$a_n = \frac{n}{2^n}.$$

Hints: In parts (b) and c) find a nicer function $f(n)$ with $|a_n| < f(n)$. Most of part (d) has already been seen on the Week 2 sheet.

Question 3: Which of the following sequences converge and to what value? In each case you should properly justify your answers, making use of the formal definition of convergence to a limit, as we have been doing in class.

(a)
$$\left(1 + \frac{(-1)^n}{n}\right)_{n\geq 1};$$

(b)
$$\left(1 + \frac{3n^2 + n}{2n^2}\right)_{n\geq 1};$$

(c)
$$(1 + (-1)^n)_{n\geq 1};$$

(d)
$$\left(\frac{n + 4(-1)^n}{2n}\right)_{n\geq 1}.$$

Question 4* (a) Let $x > 0$. Using the binomial theorem (or otherwise) prove that for all $n \geq 1$, one has $(1 + x)^n \geq 1 + nx$.
(b) By taking $x = \frac{y}{n}$ in (a), deduce that for all $y > 0$ and $n \geq 1$, $(1 + y)^{\frac{1}{n}} \leq 1 + \frac{y}{n}$.
(c) Hence show that for fixed $c > 1$, one has $c^{1/n} \to 1$ as $n \to \infty$.