MATH10242 Sequences and Series: Exercises 11, for Week 12 Tutorials
Revision: Extra Exercises

These questions are very much in the style of typical final exam questions, plus some additional, maybe harder, parts.

Question 1:
(a) Define what it means for a sequence $(a_n)_{n\geq 1}$ to tend to $-\infty$.

(b) Define what it means for a sequence $(a_n)_{n\geq 1}$ to be bounded below.

(c) The following statement is not correct; modify it to a correct statement.
"Every sequence contains a convergent subsequence."
(Adding "It is not true that" at the beginning is not what I have in mind.)

(d) Some students seem to believe that every sequence is convergent or tends to $+\infty$ or tends to $-\infty$ or switches between a finite number of values. Give an example of a sequence which has none of these properties.

Question 2:
(a) Fix $\varepsilon > 0$. Find a natural number N such that

$$\left| \frac{2n^3 - \ln n}{n(n-1)^2 + 1} - 2 \right| < \varepsilon$$

for all $n \geq N$.
What have you (if you managed to answer the question) just shown about the sequence

$$a_n = \frac{2n^3 - \ln n}{n(n-1)^2 + 1}?$$

You might make use of the fact that for all $c > 0$ we have $\ln x < x^c/c$ for all $x \geq 1$.

(b) Fix a real number K. Find a natural number N such that $\ln n - n < K$ for all $n \geq N$.

What does that prove?

Question 3: Find the limits of the following sequences.

(a) $\left(\dfrac{3^n - n^4}{2^n + n!} \right)_{n\geq 1}$

(b) $\left((n + n^{1/2})^{1/2} - n^{1/2} \right)_{n\geq 1}$

Question 4: Using L'Hôpital's Rule or otherwise, find

(a) $\lim_{n\to\infty} \dfrac{\ln(4n^{1/3} - 2)}{\ln(n+1)}$ and

(b) $\lim_{n\to\infty} \dfrac{\ln(e^{e^n} - n^6)}{n! - n^{10}}$.

Question 5: Determine whether the following series converge. In each case you should briefly justify your answer (for example by saying what test you are using).

(a) $\sum_{n=1}^{\infty} n^{10}e^{-n}$ (b) $\sum_{n=1}^{\infty} \frac{n^n}{e^n}$ (c) $\sum_{n=1}^{\infty} \frac{e^n}{e^{n^2}}$

(d) $\sum_{n=1}^{\infty} \frac{2^n n^3}{3^n}$ (e) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{\frac{3}{2}}}$ (f) $\sum_{n=1}^{\infty} \tan(\frac{\pi}{2} - \frac{1}{n})$

(g) $\sum_{n=1}^{\infty} \frac{3n^4}{n(n+e)^2}$ (h) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^2}$ (i) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Question 6: Using partial fractions or otherwise, find

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}.$$

(b) Show that the following series converge and show (use partial fractions) that they have the same sum.

$$\sum_{n=1}^{\infty} \frac{1}{2n(2n+1)} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{n}$$

Question 7: (a) Define what it means for a sequence $(a_n)_{n\geq 1}$ to (i) converge to a limit ℓ, (ii) tend to $+\infty$ as $n \to \infty$ [the original wording said "converge to ∞" but it's not a good idea to use the word "converge" next to a divergent series].

(b) Given a sequence $(a_n)_{n\geq 1}$, define a new sequence $(a_n^*)_{n\geq 1}$ by

$$a_n^* = \frac{1}{2}(a_n + a_{n+1}).$$

Prove direct from your definitions above that (i) if $a_n \to \ell$ as $n \to \infty$ then $a_n^* \to \ell$ as $n \to \infty$, (ii) if $a_n \to \infty$ as $n \to \infty$ then $a_n^* \to \infty$ as $n \to \infty$.

(c) Show, by producing suitable examples, that the converse of each of (b)(i) and (b)(ii) is false.

Question 8: Let b be a positive real number and define the sequence $(a_n)_{n\geq 1}$ inductively by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = \frac{a_n}{a_n + b} \text{ for } n \geq 1.$$

(a) Prove by induction on n that $a_n > 0$ for all n.

(b) Prove that if $0 < b < 1$ then $a_n > 1 - b$ for all n.

(c) Deduce that, if $b > 0$, then the sequence $(a_n)_{n\geq 1}$ is a decreasing sequence and, by quoting a suitable theorem, deduce that it converges.

(d) Prove that if $0 < b < 1$ then $a_n \to 1 - b$ as $n \to \infty$.

(e) Calculate $\lim_{n\to\infty} a_n$ in the case that $b \geq 1$.

Question 9:

(a) Find the radius of convergence for the series

(i) $\sum_{n=1}^{\infty} \frac{\sqrt{(2n)!}}{n!} x^n$ (ii) $\sum_{n=1}^{\infty} \frac{\sqrt{(2n)!}}{(n+1)!} x^n$

(b) Find the interval of convergence for the series

(i) $\sum_{n=1}^{\infty} \frac{x^n}{n}$ (ii) $\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt{n}}$ (iii) $\sum_{n=1}^{\infty} \frac{(-x)^n}{7n-5}$

Question 10 What is

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}}}?$$

Hint Look upon this as the limit (if it exists) of

$$\frac{1}{1}, \ \frac{1}{1+\frac{1}{1}} = \frac{1}{2}, \ \frac{1}{1+\frac{1}{1+\frac{1}{1}}} = \frac{2}{3}, \ \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}} = \frac{3}{5}, \ldots$$

Continuing,

$$1, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34} \ldots$$

We see here two subsequences

$$1, \ \frac{2}{3} \approx 0.66.., \ \frac{5}{8} = 0.625, \ \frac{13}{21} \approx 0.619.., \ \frac{34}{55} \approx 0.618.., \ldots.$$

and

$$\frac{1}{2} = 0.5, \ \frac{3}{5} = 0.6, \ \frac{8}{13} \approx 0.615.., \ \frac{21}{34} \approx 0.617..., \ldots$$

If we denote our sequence by $(a_n)_{n\geq 1}$ the evidence suggests that $(a_{2n})_{n\geq 1}$ is an increasing sequence and $(a_{2n-1})_{n\geq 1}$ is a decreasing sequence.

Our sequence $(a_n)_{n\geq 1}$ can be defined inductively by $a_1 = 1$ and

$$a_{n+1} = \frac{1}{1+a_n},$$

for $n \geq 1$.

Let ℓ be the positive root of $x^2 + x - 1 = 0$.

i. Prove that $a_{2n} \leq \ell$ and $a_{2n-1} \geq \ell$ for all $n \geq 1$.

ii. Prove that $(a_{2n})_{n\geq 1}$ is an increasing sequence and $(a_{2n-1})_{n\geq 1}$ is a decreasing sequence.

iii. Show that $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n-1} = \ell$.