MATH10242 Sequences and Series: Exercises 10, for Week 11 Tutorials

Be sure that you can do examples like those in Question 1.

Question 1: Find the radius of convergence R of the following power series.
In parts (i) and (ii), what is the *interval of convergence* of the given power series?

$$(i)\sum_{n\geq 1}\frac{x^n}{8^n} \qquad (ii)\sum_{n\geq 1}\frac{(-x)^n}{4n+1}, \qquad (iii)\sum_{n\geq 1}\frac{(2n)!}{(n!)^2}x^n, \qquad (iv)\sum_{n\geq 1}\frac{n^n}{n!}x^n,$$

$$(v)\sum_{n\geq 1} n!\cdot x^n \qquad (vi)\sum_{n\geq 1}\frac{\sqrt{(2n)!}}{n!}x^n.$$

[You will need to use the formula for $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ from a previous Exercise Sheet.]

Question 2: Let $r > 0$. Using Question 1(i) as a guide, find a series $\sum_{n=1}^{\infty} a_n x^n$ with radius of convergence r.

Question 3: Let $\sum_{n\geq 1} a_n$ be a series. We define two new series $\sum_{n\geq 1} a_n^+$, consisting of all the positive terms of the original series and and $\sum_{n\geq 1} a_n^-$, consisting of all the negative terms. To be specific, set

$$a_n^+ = \frac{a_n + |a_n|}{2} \qquad \textit{and} \qquad a_n^- = \frac{a_n - |a_n|}{2},$$

and notice that if $a_n > 0$ then $a_n^+ = a_n$ and $a_n^- = 0$. Conversely, if $a_n < 0$ then $a_n^- = a_n$ and $a_n^+ = 0$.

(a) Prove that, if $\sum_{n\geq 1} a_n$ is absolutely convergent, then both $\sum_{n\geq 1} a_n^+$ and $\sum_{n\geq 1} a_n^-$ are convergent. Moreover, prove that

$$\sum_{n\geq 1} a_n = \sum_{n\geq 1} a_n^+ + \sum_{n\geq 1} a_n^-.$$

(b*) Prove that, if $\sum_{n\geq 1} a_n$ is only conditionally convergent, then both $\sum_{n\geq 1} a_n^+$ and $\sum_{n\geq 1} a_n^-$ are divergent.