MATH10242 Sequences and Series: Exercises 9, for Week 10 Tutorials

It is important to practice deciding which test looks most likely to work for a particular example.

Question 1: Use the Integral Test to test the series below for convergence or divergence. In (i) (ii) and (iii) is there some other test that would also work?

$$(i)\ \sum_{n\geq 1}\frac{1}{n^2+1},\qquad (ii)\ \sum_{n\geq 1}\frac{n}{n^2+1},\qquad (iii)\ \sum_{n\geq 1}n^2e^{-n},\qquad (iv)\ \sum_{n\geq 2}\frac{1}{n(\ln n)^p},\ \text{for } p>1.$$

Remark: In the next question, you are given a number of different series and you should find out if they converge or not. The key step in any such question is: given a series $\sum_{n\geq 1} a_n$ what test should we use?

Expanding on 9.2.11, you could try following:

a) Is $\lim_{n\to\infty} a_n = 0$? If not, then it diverges by 8.1.2.

b) Does the Integral Test apply? If so that should work.

c) If there are exponentials or factorials, then the Ratio Test should work.

d) Try to use the Comparison Test; looking for the fastest-growing term(s) will tell you what you should compare your series to.

As always, it's the eventual behaviour of the terms that determines convergence/non-convergence, so you can ignore the first few terms for that issue (of course, they make a difference to the value of the series if it does converge).

Question 2: Test the series below for convergence or divergence

(i) $\displaystyle\sum_{n\geq 1}\sin\left(\frac{1}{n^2}\right)$ [Hint: first show that $\sin(x) < x$ for all $x > 0$.]

(ii) $\displaystyle\sum_{n\geq 1}\tan\left(\frac{1}{n^2}\right)$ (iii) $\displaystyle\sum_{n\geq 1}\cos\left(\frac{1}{n^2}\right)$ (iv) $\displaystyle\sum_{n\geq 1}\frac{n!}{(n+1)!}$ (v) $\displaystyle\sum_{n\geq 1}\frac{n!}{(n+2)!}$

(vi) $\displaystyle\sum_{n\geq 1}n^{-2}\cos(1/n)e^{\sin(1/n)}$ (vii) $\displaystyle\sum_{n\geq 2}n^3e^{-n^4}$ (viii) $\sum_{n\geq 2}\frac{1}{n(\ln n)}$ (ix) $\displaystyle\sum_{n\geq 2}\frac{1+\ln n}{n(\ln n)^2}$

Question 3: Given a convergent series $\sum_{n=1}^{\infty} a_n$ and sequence $(b_n)_{n\geq 1}$ with $|b_n| \leq |a_n|$ for all $n \geq 1$ is it true that $\sum_{n=1}^{\infty} b_n$ converges? Justify your answer.

Question 4: Test the series below for convergence and for absolute convergence. Which are conditionally convergent?

For this question, try first just writing down the answer with only a brief reason why it is true—for example if one had the series $\sum(n^2+1)/(n^4+3n^2)$ you might write "absolutely convergent and hence convergent by comparison with $\sum 1/n^2$". Then you can check some or all of them by doing the details.

(*i*) $\displaystyle\sum_{n\geq 1}(-1)^n\left(\frac{n+1}{n+2}\right)$, (*ii*) $\displaystyle\sum_{n\geq 1}(-1)^n\left(\frac{n+1}{n^2+2}\right)$, (*iii*) $\displaystyle\sum_{n\geq 1}(-1)^n\left(\frac{n+1}{n^3+1}\right)$

(*iv*) $\displaystyle\sum_{n\geq 1}(-1)^n\frac{\cos(n)}{n^2}$ (*iv*) $\displaystyle\sum_{n\geq 1}\frac{1}{(-2)^n}$.

Question 5: (The bouncing ball). In Exercise 9.1.10 of the notes, we had a bouncing ball, dropped from a height of 1 meter and that each time it bounced it rose to a height of (2/3) times the previous height. As we saw, it travels a total of 5 metres.
Question, does it keep bouncing for ever, or does it stop? (We know in practice that it stops, but maybe that is a consequence of friction, etc.)

(a) To answer this first show that a ball which is dropped (ie. which has an initial speed of 0) travels $9.8t^2/2$ meters in t seconds. Hence if it drops from a height of h metres it reaches the ground in $\sqrt{h}/\sqrt{4.9}$ seconds.

(b) Now show that it stops bouncing after

$$\frac{1}{\sqrt{4.9}}\left(1+2\sqrt{\frac{2}{3}}\left(\frac{1}{1-\sqrt{2/3}}\right)\right)$$

seconds. This is about 4.8 seconds.