Part II

Series

Chapter 8

Introduction to Series

In this chapter we give a rigorous definition of *infinite* sums of real numbers. We are interested in expressions of the form

$$\sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots + a_n + \cdots$$

and need to make sense of such expressions. This is closely related to sequences since this "infinite sum" is (by definition! as you will see) the limit of the *sequence of partial sums* (s_n) where

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots a_n.$$

You have probably seen that

$$\sum_{i=0}^{\infty} \frac{1}{2^i} = 2.$$

To see that: an easy induction shows that

$$s_n = \sum_{i=0}^{n} \frac{1}{2^i} = 2 - \frac{1}{2^n},$$

then the limit of the sequence $(s_n)_n$ of partial sums is 2.

However, there are some surprising subtleties in these series. One of the basic tricky ones is the *Harmonic Series*:

$$\sum_{i=1}^{\infty} \frac{1}{n}$$

which equals ∞. That might seem counter-intuitive but, to see it, collect terms together to get:

$$\sum_{i=1}^{\infty} \frac{1}{n} = 1 + \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{3}+\tfrac{1}{4}\right) + \left(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}\right) + \cdots$$
$$\geq 1 + \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right) + \cdots . \tag{8.1}$$

So this gives an infinite sum of halves, which is therefore infinite. In contrast, we will see that

$$\sum_{i=1}^{\infty} \frac{1}{n^{1.001}} < \infty.$$

So, we are going to need some new techniques!

8.1 The Basic Definitions

For real numbers a_n, an **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n \quad \text{(also written } a_1 + a_2 + a_3 + \cdots + a_n + \cdots\text{)}.$$

This may also be written $\sum_{n\geq 1} a_n$ or even just $\sum a_n$.

Given such a series, we form **the sequence of partial sums** $(s_n)_{n\geq 1}$ defined by setting

$$s_n = a_1 + a_2 + \cdots a_n = \sum_{i=1}^{n} a_i.$$

If $s_n \to s$ as $n \to \infty$, we then say that the series $\sum_{n=1}^{\infty} a_n$ is *convergent with sum* s, and write $\sum_{n=1}^{\infty} a_n = s$.

If there is no real number s with this property, that is if the sequence (s_n) does not have a limit, then we say that the series $\sum_{n=1}^{\infty} a_n$ is **divergent**. If the series is divergent but $\lim_{n\to\infty} s_n = +\infty$ then we say that $\sum_{n=1}^{\infty} a_n = +\infty$ (similarly with $-\infty$).

The Harmonic Series, revisited. Let's first check that the harmonic series (8.1) really does have sum ∞. So, what we have seen is that for any K there exists N (in fact $N = 2^{2K}$ works) such that $\sum_{n=1}^{N} 1/n$ gives (more than) a sum of $2K$ copies of $1/2$. In other words, if $m \geq N$ then

$$s_m = \sum_{n=1}^{m} \frac{1}{n} \geq \sum_{n=1}^{N} \frac{1}{n} \geq K.$$

Which is just what one needs to prove to see that $\sum_{i=1}^{\infty} 1/n = +\infty$.

Remark. Sometimes (as in the next example), it is more convenient to start the series/sequence at 0, thus looking at series of the form $\sum_{n=0}^{\infty} = a_0 + a_1 + \cdots$.

Example 8.1.1 (Geometric series). Let r be a real number with $|r| < 1$. Then the series $\sum_{n=0}^{\infty} r^n$ converges with sum $1/(1-r)$.

Proof. Here, $a_n = r^n$ and

$$s_n = a_0 + a_1 + \cdots + a_n = 1 + r + \cdots + r^n.$$

So

$$rs_n = r + r^2 + \cdots + r^{n+1}.$$

Subtracting we obtain

$$(1-r)s_n = 1 - r^{n+1}$$

because all the other terms cancel. Hence

$$s_n = \frac{1 - r^{n+1}}{1 - r}.$$

Now since $-1 < r < 1$, $r^{n+1} \to 0$ as $n \to \infty$. (Use 4.1.2.) Hence

$$s_n = \frac{1 - r^{n+1}}{1 - r} \to \frac{1 - 0}{1 - r} = \frac{1}{1 - r}$$

(by AoL). So, by definition,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}.$$

■

Remark: This proof only works for $|r| < 1$. It might be tempting to put in other values of r since the right hand side is well defined for any r with $r \neq 1$. But this gives nonsense, e.g.

$$\sum_{n=0}^{\infty} 2^n = \frac{1}{1 - 2} = -1$$

is definitely *not* correct.

Example 8.1.2 (Using partial fractions). Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

converges to 1.

Proof Here $a_n = 1/n(n+1)$ and

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}.$$

A trick that sometimes works is to use partial fractions on the terms. Here we use the identity

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

valid for $x \neq 0, -1$. Hence

$$s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right).$$

Now all the terms cancel apart from the first and the last, so we obtain

$$s_n = 1 - \frac{1}{n+1} \to 1 - 0 = 1 \text{ as } n \to \infty.$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

■

Remark: It is very important in Example 8.1.2 that one only rearranges terms in a *finite* sum rather than rearranging terms in the infinite sum $\sum_{n=1}^{\infty} 1/n(n+1)$. The reason is that, with infinite sums, you can get all sorts of strange different answers by arranging things in different ways—the details can be found in Section 12.2.

It is quite rare for one to be able to find an exact formula for the nth partial sum (i.e. for the number s_n) as we did in these two examples. So, just as we did with the study of sequences, we need to build up a stock of general theorems that will help us to tell when a series does and does not converge without explicitly computing the partial sums. Our first result along these lines states that a series cannot converge unless the terms (i.e. the a_n) form a null sequence. Note, however, that the *converse* of this statement is false in general, as we saw with $\sum_{n=1}^{\infty} 1/n$. Here is a similar type of example.

Example 8.1.3 (of non-convergence). Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges.

Proof Here, $a_n = 1/\sqrt{n}$ and we have shown (see Example 3.2.3) that $(a_n)_{n\geq 1}$ is a null sequence. However, the series $\sum_{n=1}^{\infty} a_n$ does not converge. For

$$s_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}},$$

from which we deduce that

$$s_n \geq (\text{number of terms}) \cdot (\text{smallest term}) = n \cdot \frac{1}{\sqrt{n}} = \sqrt{n}.$$

Thus $s_n \geq \sqrt{n}$ for all $n \in \mathbb{N}$, so $s_n \to +\infty$ as $n \to \infty$ by the Infinite Sandwich Rule (since $\sqrt{n} \to +\infty$ as $n \to \infty$). Hence, by definition, the series $\sum_{n=1}^{\infty} 1/\sqrt{n}$ does not converge. ■

Theorem 8.1.4. (The n^{th} term Test.) *If $\sum_{n=1}^{\infty} a_n$ is a convergent series, then $(a_n)_{n\geq 1}$ is a null sequence.*

Proof. Suppose that $\sum_{n=1}^{\infty} a_n = s$. Let $s_n = a_1 + \cdots + a_n$ be the nth partial sum. So $s_n \to s$ as $n \to \infty$. Hence $\lim_{n\to\infty} s_{n-1} \to s$ as well (see Lemma 4.1.3), and so by AoL,

$$\lim_{n\to\infty} (s_n - s_{n-1}) = s - s = 0.$$

But $s_n - s_{n-1} = a_n$. So $a_n \to 0$ as $n \to \infty$, i.e. $(a_n)_{n\geq 1}$ is a null sequence. ■

Theorem 8.1.5. (Algebra of Infinite Sums)

(i) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both convergent series, with sums A and B respectively, then the series $\sum_{n=1}^{\infty}(a_n + b_n)$ is also convergent with sum $A + B$.

(ii) If $\sum_{n=1}^{\infty} a_n$ is a convergent series with sum A, and λ is any real number, then the series $\sum_{n=1}^{\infty} \lambda a_n$ is convergent with sum λA.

Proof. (i) For $n \in \mathbb{N}$, let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$ be the partial sums and similarly let $u_n = \sum_{k=1}^{n}(a_k + b_k)$ be the partial sums for the series $\sum\limits_{n=1}^{\infty}(a_n + b_n)$.

Then, rearranging terms in a finite sum as we might,

$$u_n = \left(\sum_{k=1}^{n} a_k\right) + \left(\sum_{k=1}^{n} b_k\right) = s_n + t_n \quad \text{for } n \in \mathbb{N}.$$

But $s_n \to A$ and $t_n \to B$ as $n \to \infty$ by the definition of convergence of the original series. Thus, by AoL for sequences, $u_n \to A + B$ as $n \to \infty$, i.e.

$$\sum_{n=1}^{\infty}(a_n + b_n) = A + B,$$

as required.

The proof of (ii) is left as an exercise. ■

Chapter 9

Series with Non-Negative Terms

In this chapter we establish some key facts about series with non-negative terms. The theory of such series is very much easier than the general case where negative terms are allowed.

9.1 The Basic Theory of Series with Non-Negative Terms

Theorem 9.1.1. *Suppose that* $a_n \geq 0$ *for all* $n \in \mathbb{N}$ *and let* $s_n = \sum_{k=1}^{n} a_k$. *Then the series* $\sum_{n=1}^{\infty} a_n$ *is convergent if and only if the sequence* $(s_n)_{n\geq 1}$ *is bounded above.*

Proof. Note that $s_{n+1} = s_n + a_{n+1} \geq s_n$, since $a_{n+1} \geq 0$. Hence the sequence $(s_n)_{n\geq 1}$ is increasing. So, by the Monotonic Convergence Theorem, if $(s_n)_{n\geq 1}$ is bounded above, then it converges and thus, by definition, the series $\sum_{n=1}^{\infty} a_n$ is convergent.

Conversely, if $\sum_{n=1}^{\infty} a_n$ is convergent, then (s_n) is convergent (definition!) and so Theorem 2.3.9 says that $(s_n)_{n\geq 1}$ is bounded above. ■

Theorem 9.1.2 (The Comparison Test)**.** *If* $0 \leq a_n \leq b_n$ *for all* $n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} b_n$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Equivalently, if $0 \leq a_n \leq b_n$ *for all* $n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} a_n$ *diverges, then* $\sum_{n=1}^{\infty} b_n$ *diverges.*

Proof. Let $s_n = a_1 + a_2 + \cdots + a_n$ and $t_n = b_1 + b_2 + \cdots + b_n$. Since $a_n \leq b_n$ for all n it follows that $s_n \leq t_n$ for all n. But as $\sum_{n=1}^{\infty} b_n$ is a convergent series of non-negative terms there is, by Theorem 9.1.1, some $M > 0$ such that $t_n \leq M$ for all n. Hence $s_n \leq M$ for all n, and so, again by 9.1.1, $\sum_{n=1}^{\infty} a_n$ is convergent. ■

Exercise 9.1.3. Let $(a_n)_{n\geq 1}$ be any sequence and let N be any positive integer. Show that the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent if and only if the series $\sum_{n\geq N} a_n$ (which is the same as the series $\sum_{n=1}^{\infty} a_{n+N-1}$) is convergent.

Remark: This exercise is done in the Exercise sheet for Week 9, and essentially means that for any given test, you need only apply the test for "large n". Let's make this precise with:

Slightly Improved Comparison Test: *Let $N \in \mathbb{N}$. If $0 \leq a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

Proof. By (9.1.3),

$$\begin{aligned}
\sum_{n=1}^{\infty} b_n \quad \text{converges} \quad &\Rightarrow \sum_{n\geq N} b_n = \sum_{m\geq 1} b_{N+m-1} \quad \text{converges} \\
&\Rightarrow \sum_{n\geq N} a_n = \sum_{m\geq 1} a_{N+m-1} \quad \text{converges (by the comparison test)} \\
&\Rightarrow \sum_{n\geq 1} a_n \quad \text{converges (by 9.1.3 again).}
\end{aligned}$$

■

Example 9.1.4. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent.

Proof We have that

$$\frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)}$$

for all $n \in \mathbb{N}$, and hence by Example 8.1.2 and the Comparison Test, $\sum_{n=1}^{\infty} 1/(n+1)^2$ is convergent. Since this is just the series $\sum_{n\geq 2} 1/n^2$, the convergence of the series $\sum_{n=1}^{\infty} 1/n^2$ follows from Exercise 9.1.3. ■

This example shows the value of the Comparison Test: it was straightforward to calculate the partial sums of the series $\sum_{n=1}^{\infty} 1/n(n+1)$ and hence calculate explicitly the sum of this series (thereby establishing its convergence). But it is impossible to give a neat formula for the partial sums of the series $\sum_{n=1}^{\infty} 1/n^2$, so we resort to comparing its terms with those of the series $\sum_{n=1}^{\infty} 1/n(n+1)$ in order to establish its convergence.

In fact, as mentioned in the introduction to this course,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

but the proof of this requires methods from next year's complex analysis course.

Example 9.1.5. For any $p \geq 2$, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent. To see this we simply observe that for all $n \in \mathbb{N}$, $1/n^p \leq 1/n^2$ (since $p \geq 2$) and apply the Comparison Test to the previous example.

Similarly we have:

Example 9.1.6. For any $p \leq 1$ the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is divergent.

Proof For suppose that $p \leq 1$ and, for a contradiction, that $\sum_{n=1}^{\infty} 1/n^p$ is convergent. Since $1/n \leq 1/n^p$ for all $n \in \mathbb{N}$ we would have, by the Comparison Test, that $\sum_{n=1}^{\infty} 1/n$ is convergent. But this contradicts Equation 8.1. ■

Remark: We've left the case $1 < p < 2$ open but, later, we will show that $\sum_{n=1}^{\infty} 1/n^p$ converges if $p > 1$ but diverges if $p \leq 1$.

We now come to an important test for the convergence of series.

Theorem 9.1.7 (The Ratio Test for series.). *Suppose that $a_n > 0$ for all $n \in \mathbb{N}$ and assume that*

$$\frac{a_{n+1}}{a_n} \to l$$

as $n \to \infty$.

(i) If $l < 1$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $l > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Remark: If $l = 1$, no conclusion can be drawn: For example, if $a_n = 1/n^2$ or $a_n = 1/n$, then in both cases we have that $a_{n+1}/a_n \to 1$ as $n \to \infty$. But in the first case $\sum_{n=1}^{\infty} a_n$ converges, whereas in the second case it diverges.

Proof. (i) Suppose that $l < 1$. We want to compare our series to the geometric series $\sum r^n$ for some r. We need a little room to manoeuvre and taking $l = r$ wouldn't give us that, so we will take a slightly bigger number for r.

So, choose r so that $l < r < 1$ (e.g. take $r = 1 + l/2$ to be the average of l and 1) and set $\varepsilon = r - l$. Then $\varepsilon > 0$ and we may choose $N \in \mathbb{N}$ so that

$$l - \varepsilon < \frac{a_{n+1}}{a_n} < l + \varepsilon$$

for all $n \geq N$.

But $\varepsilon = r - l$ says that $l + \varepsilon = r$ and so, as $a_n > 0$ we get $0 < a_{n+1} < r \cdot a_n$ for $n \geq N$. In particular

$$a_{N+1} < r \cdot a_N, \qquad \textit{and} \qquad a_{N+2} < r \cdot a_{N+1} < r^2 \cdot a_N$$

and so on. So (by induction) we obtain

$$\forall n \geq N, \quad 0 < a_n < r^{n-N} \cdot a_N$$

Collecting terms we see that

$$\sum_{n \geq N} a_n \;\leq\; \sum_{n \geq N} r^{n-N} \cdot a_N \;=\; \sum_{n \geq 0} r^n \cdot a_N \;=\; a_N \sum_{n \geq 0} r^n. \qquad (*)$$

But as $0 < r < 1$ it follows from Example 8.1.1 that $\sum_{n \geq 0} r^n$ converges. Now we are basically done: by The Algebra of Infinite Sums 8.1.5(ii), $a_N \sum_{n \geq 0} r^n$ converges and so from (*) and the Slightly Improved Comparison Test, $\sum_{n \geq 1} a_n$ converges, as required.

(ii) Now suppose that $l > 1$. In this case we choose r so that $1 < r < l$ and, by following the same procedure as above, we obtain an $N \in \mathbb{N}$ such that for all $n \geq N$ we have

$$a_n > r^{n-N} \cdot a_N = r^n \cdot \left(\frac{a_N}{r^N} \right).$$

But, since $r > 1$, this implies that $a_n \to \infty$ as $n \to \infty$ and, in particular, it implies that $(a_n)_{n \geq 1}$ is not a null sequence. So $\sum_{n=1}^{\infty} a_n$ diverges by Theorem 8.1.4. ∎

Example 9.1.8. Show that the series

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

converges.

Proof Let $a_n = n^2/2^n$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2} \cdot \left(1 + \frac{1}{n}\right)^2.$$

So, by AoL,

$$\frac{a_{n+1}}{a_n} \to \frac{1}{2} \cdot (1+0)^2 = \frac{1}{2} \qquad \text{as } n \to \infty.$$

Since $1/2 < 1$, it follows from the Ratio Test that $\sum_{n=1}^{\infty} n^2/2^n$ converges. ■

Example 9.1.9. Show that the series Let x be any positive real number and consider the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

converges for all $x > 0$.

Proof Here, $a_n = x^n/n!$, so

$$\frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \frac{x}{n+1}.$$

But $x/(n+1) \to 0$ as $n \to \infty$ and $0 < 1$. So by the Ratio Test, $\sum_{n=0}^{\infty} x^n/n!$ converges. ■

Example 9.1.10. (The Bouncing Ball.) Suppose that a rubber ball is dropped from a height of 1 metre and that each time it bounces it rises to a height of (2/3) of the previous height. How far does it travel before it stops bouncing (and yes it does stop)?

Solution: First it drops $1m$. Then it rises up and drops $2/3m$. Then it rises up and drops $(2/3)^2 m$, etc etc. So the total distance travelled is

$$\begin{aligned} &1 + 2 \times \tfrac{2}{3} + 2 \times \left(\tfrac{2}{3}\right)^2 + 2 \times \left(\tfrac{2}{3}\right)^3 + \cdots \\ &= 1 + 2 \times \tfrac{2}{3}\left(1 + \tfrac{2}{3} + \left(\tfrac{2}{3}\right)^2 + \cdots\right) = 1 + \tfrac{4}{3} \cdot \frac{1}{1 - 2/3} = 5. \end{aligned}$$

■

9.2 The Integral Test

We consider a function $f : [1, \infty) \to \mathbb{R}$ which is:

(i) positive (at least, non-negative), so $f(x) \geq 0 \;\; \forall x \geq 1$;

(ii) decreasing, so $f(x) \leq f(y) \;\; \forall \; x \geq y \geq 1$;

(iii) continuous.

Here continuity (meaning being continuous) is a condition you may have seen before, but will be made precise in your analysis course next year. For the record (although you need not remember this) the definition is as follows. A function f is **continuous on** $X = [1, \infty)$ if for all $x \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that if $y \in \mathbb{R}$ with $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. That is,

$$\forall x \in X, \forall \varepsilon > 0, \exists \delta > 0 : \forall y \in \mathbb{R}, |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Roughly speaking it means that "there are no breaks in the graph of f". This condition rules out functions such as

$$f(x) = \begin{cases} 1 & \text{if } x \leq 2, \\ \frac{1}{2} & \text{if } x > 2. \end{cases}$$

(In this example, try taking $x = 2$ and $\varepsilon = 1/4$.)

Examples of functions $f(x)$ satisfying (i), (ii) and (iii) are

$$\frac{1}{x}, \quad \frac{1}{x^2}, \quad \frac{1}{x \ln(1 + x)}, \quad 2^{-x}.$$

Theorem 9.2.1 (The Integral Test). *Let $f : [1, \infty) \to \mathbb{R}$ be a function satisfying the three conditions above. Then the series*

$$\sum_{n=1}^{\infty} f(n)$$

converges if and only if the sequence

$$\left(\int_1^n f(x) dx \right)_{n \geq 1}$$

converges as $n \to \infty$.

This in turn happens if and only if the sequence $\left(\int_1^n f(x)dx\right)_{n\geq 1}$ *is bounded.*

Before giving the proof here are some examples of how the test works.

Example 9.2.2. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

We shall show that it diverges.

Proof Let $f : [1, \infty) \to \mathbb{R}$ be the function $f(x) = 1/x$, which clearly satisfies our three conditions.

Now

$$\int_1^n f(x)dx = \int_1^n \frac{dx}{x} = [\ln x]_1^n = \ln n - \ln 1 = \ln n.$$

But (as in Example 7.1.3) $(\ln n)_{n\geq 1}$ is a divergent sequence, so by the Integral Test, $\sum_{n=1}^{\infty} 1/n$ is a divergent series. ■

Here is the important example which was left partly unresolved in the previous chapter:

Example 9.2.3. If p is a real number with $p > 1$, then

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is a convergent series.

it Proof Here we take $f(x) = 1/x^p$. This clearly satisfies the three conditions for the Integral Test to be applicable. We have

$$\int_1^n f(x)dx = \int_1^n x^{-p}dx = \left[\frac{x^{-p+1}}{(-p+1)}\right]_1^n = \frac{n^{-p+1}}{-p+1} - \frac{1}{-p+1} = \frac{1}{p-1}\left(1 - \frac{1}{n^{p-1}}\right).$$

Now since $p > 1$, we have that $1/n^{p-1} \to 0$ as $n \to \infty$. So

$$\int_1^n \frac{1}{x^p}dx \to \frac{1}{p-1}(1-0) = \frac{1}{p-1} \quad \text{as } n \to \infty.$$

In particular, the sequence $\left(\int_1^n 1/x^p dx\right)_{n\geq 1}$ is convergent (with limit $1/(p-1)$). Hence, by the Integral Test, the series $\sum_{n=1}^{\infty} 1/n^p$ is convergent (for $p > 1$). ■

Exercise 9.2.4. Use the Integral test to prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is divergent if $p < 1$. (We also proved this in Example 9.1.6.)

Proof of the Integral Test: First of all, it is clear that

$$\int_1^n f(x)dx \leq \int_1^{n+1} f(x)dx$$

and so the sequences of integrals is an increasing sequence. Hence the second paragraph of the Theorem is nothing more than the Monotone Convergence test.

So we look at the first assertion. The basic idea is that the integral $\int_{x=1}^{n} f(x)dx$ defines the area under that curve for $1 \leq x \leq n$. This we can approximate by summing the areas of a series of rectangles with width 1. For $x : n \leq x \leq n+1$ we have, since f is a decreasing function, that $f(n) \geq f(x) \geq f(n+1)$. Integrate this over the interval $[n, n+1]$ to get

$$\int_n^{n+1} f(n)\, dx \geq \int_n^{n+1} f(x)dx \geq \int_n^{n+1} f(n+1)\, dx.$$

Since the integral of a constant equals the constant times the length of the interval, in this case 1, we get

$$f(n) \geq \int_n^{n+1} f(x)\, dx \geq f(n+1)\,.$$

This holds for all real n though we only use it for integers n. In fact, adding these inequalities for $n = 1, 2, 3, ..., N-1$, the integrals all combine as one integral from 1 to N with

$$\sum_{n=1}^{N-1} f(n) \geq \int_1^N f(x)\, dx \geq \sum_{n=1}^{N-1} f(n+1) = \sum_{n=2}^{N} f(n) \qquad (**).$$

After this preparatory work we now prove Theorem 9.2.1.

($\Rightarrow$): Suppose that the series $\sum_{n=1}^{\infty} f(n)$ converges. Then the partial sums are bounded (by 9.1.1, since the terms $f(n)$ are positive). So there is a positive real number M such that for all $N \in \mathbb{N}$ we have that

$$\sum_{n=1}^{N-1} f(n) \leq M.$$

Hence by (∗∗),

$$\int_1^N f(x)dx \leq M$$

for all $N \in \mathbb{N}$ and so the sequence $\left(\int_1^n f(x)dx\right)_{n\geq 1}$ is a bounded sequence.

But since f is a positive function, an increase in the range over which we integrate f will result in a larger value for the integral. This implies that the sequence $\left(\int_1^n f(x)dx\right)_{n\geq 1}$ is an increasing sequence. Thus, by the Monotone Convergence Theorem, it is a convergent sequence as required.

(⇐): Suppose that the sequence $\left(\int_1^n f(x)dx\right)_{n\geq 1}$ converges. Then it is bounded (by Theorem 2.3.9). Let L be a bound for it. So

$$\int_1^N f(x)dx \leq L$$

for all $N \in \mathbb{N}$. Hence, by (∗∗),

$$\sum_{n=2}^{N} f(n) \leq L$$

for all $N \in \mathbb{N}$. This means that the partial sums of the series $\sum_{n=2}^{\infty} f(n)$ are bounded and so, being a series of positive terms, it converges (by 9.1.1). But then so does the series $\sum_{n=1}^{\infty} f(n)$ (by Exercise 9.1.3), as required. ■

This completes the proof of the Integral Test.

Remark 9.2.5. Sometimes the integral $\int_1^n f(x)dx$ can be awkward to compute at the Left Hand End, but in that case, since it is always OK to compute integrals of the form $\int_K^n f(x)dx$ for some fixed $1 \leq K$ (and then $K \leq n$), it may be that there is a convenient choice for K.

Indeed, in view of 9.1.3, one only has to verify that that the three conditions (i), (ii) and (iii) hold for all *sufficiently large* x, i.e. for all $x \geq K$ (for some given $K \in \mathbb{N}$). The same proof shows that

$$\sum_{n=1}^{\infty} f(n) \text{ converges} \iff \sum_{n=K}^{\infty} f(n) \text{ converges} \iff \left(\int_K^n f(x)dx\right)_{n\geq K} \text{ converges.}$$

Example 9.2.6. Prove that the series

$$\sum_{n=2}^{\infty} \frac{n^2}{n^3-1}$$

diverges.

Proof: Consider the function $f : [2, \infty) \to \mathbb{R}$ defined by $f(x) = x^2/(x^3 - 1)$.

Clearly $f(x) > 0$ for $x \geq 2$ and f is continuous. To see that f is decreasing, we can either use calculus or algebra. Using calculus is easier: after simplification we have $f'(x) = -(x^4 + 2x)(x^3 - 1)^{-2}$. Clearly this is negative for $x > 1$ and so our function $f(x)$ decreases.

For an algebraic proof, suppose that $1 \leq x < y$. Then

$$\begin{aligned}
\frac{x^2}{x^3-1} \geq \frac{y^2}{y^3-1} &\iff x^2y^3 - x^2 \geq y^2x^3 - y^2 \\
&\iff y^2 - x^2 \geq y^2x^3 - x^2y^3 \\
&\iff (y-x)(y+x) \geq y^2x^2(x-y) \\
&\iff (y+x) \geq -y^2x^2
\end{aligned}$$

Since the last statement here is true we can reverse the bi-implications. This shows that f is indeed decreasing.

So we may apply the Integral Test (in the form of 9.2.5). Now

$$\int_2^n f(x)dx = \int_2^n \frac{x^2}{x^3-1}dx.$$

In order to evaluate the integral we make the substitution $u = x^3 - 1$. Thus $du = 3x^2dx$ and the new limits are $u = 7$ to $u = n^3 - 1$. Thus

$$\int_2^n f(x)dx = \frac{1}{3}\int_7^{n^3-1} \frac{du}{u} = \frac{1}{3}\left[\ln u\right]_7^{n^3-1} = \frac{1}{3}(\ln(n^3-1) - \ln 7).$$

Since $\frac{1}{3}(\ln(n^3-1) - \ln 7) \to \infty$ as $n \to \infty$, the sequence $\left(\int_2^n f(x)\right)_{n\geq 2}$ diverges and hence so does the series $\sum_{n=2}^{\infty} n^2/(n^3-1)$ by the Integral Test. ■

Remark 9.2.7. *Assume that conditions (i)–(iii) from Theorem 9.2.1 hold. Let $K \in \mathbb{N}$ (usually I would have $K = 1$.) Then integrals of the form $\int_K^\infty f(x)dx$ are called* **improper** *integrals and in more detail we have the following.*

For any $K < r < s$ we have

$$0 \leq F(r) = \int_K^r f(x)dx \leq \int_K^r f(x)dx + \int_r^s f(x)dx = \int_K^s f(x)dx = F(s),$$

and so by the Monotone Convergence Theorem either

1. *The sequence $\left(\int_K^n f(x)dx\right)_{n\geq K}$ is bounded and hence convergent, in which case we*

write its limit as $\int_K^\infty f(x)dx$ *and say that "*$\int_K^\infty f(x)dx$ *converges".*

2. *Alternatively the sequence* $\left(\int_K^n f(x)dx\right)_{n\geq K}$ *is unbounded and hence tends to infinity. In this case we say "*$\int_K^\infty f(x)dx$ *diverges" or indeed that* $\int_K^\infty f(x)dx = \infty$.

So we paraphrase Theorem 9.2.1 as saying that

$$\sum_{n=1}^\infty f(n) \text{ converges} \iff \int_1^\infty f(x) \text{ converges.}$$

(Finally, note that talking about improper integrals usually involves a somewhat different kind of limit, namely the limit of the $F(r)$ *as the real number* r *tends to infinity. Also, if the conditions (i)–(iii) do not hold, then you have to be much more careful with the definitions and properties of improper integrals.)*

Example 9.2.8. Prove that the series

$$\sum_{n=2}^\infty \frac{1}{n \ln n}$$

diverges.

Proof Let $f(x) = 1/(x \ln x)$ in the Integral Test and consider the integral

$$\int_2^n \frac{1}{x \ln x} dx.$$

In order to evaluate it we make the substitution $u = \ln x$. Then $du = dx/x$ and the new range of integration is from $u = \ln 2$ to $u = \ln n$. Thus

$$\int_2^n \frac{1}{x \ln x} dx = \int_{\ln 2}^{\ln n} \frac{du}{u} = [\ln u]_{\ln 2}^{\ln n} = \ln(\ln n) - \ln(\ln 2).$$

Now (exercise) $\ln(\ln n) \to \infty$ as $n \to \infty$ and so the sequence $\left(\int_2^n 1/(x \ln x)dx\right)_{n\geq 1}$ diverges. Hence, by the Integral Test, the series $\sum_{n=2}^\infty 1/(n \ln n)$ diverges. ∎

Exercise 9.2.9. Let $p \in \mathbb{R}$. Prove that the series

$$\sum_{n=2}^\infty \frac{1}{n(\ln n)^p}$$

converges if and only if $p > 1$. What about the series

$$\sum_{n=3}^\infty \frac{1}{n \ln n \, (\ln \ln n)^p} \ ?$$

(We take the lower limit of summation to be 3 here because $\ln\ln 2$ is negative and so $(\ln\ln 2)^p$ might not be defined.)

Final Remark: In the integral test you do have to be careful to check that conditions (i - iii) hold. The trouble is that without them it is easy to get silly counterexamples. For example, suppose we take the function $y(x) = 1 + \cos((2n+1)\pi)$; this function has been chosen so that $f(n) = 0$ for all n and $f(x) \geq 0$ for all $x \geq 0$. So, here $\sum f(n) = 0 + \cdots + 0 = 0$ is certainly convergent, whereas the integral $\int_{x=1}^{\infty} f(x)dx = \infty$.

Exercises 9.2.10. No doubt when you first saw the techniques of integration it took a while to get the intuition about which technique to use for which example. The same goes for our techniques for testing the convergence of infinite series $\sum a_n$ (and sequences). The only way to develop this intuition is to do lots of examples. Here are some examples, with partial hints about how to approach them:

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$.

Answer: Diverges by 9.1.1 as $\lim_{n\to\infty} a_n \neq 0$.

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$.

Answer: Converges. Since

$$\frac{\sqrt{n^3+1}}{3n^3+4n^2+2} \leq \frac{\sqrt{n^3+n^3}}{3n^3} \leq \frac{\sqrt{2}}{3n^{3/2}}$$

our series converges by comparison with the convergent $\sum_{n=1}^{\infty} 1/n^{3/2}$. (See CT and 9.2.3.

(3) $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$.

Answer: This converges, either by the ratio test or the integral test. By integrating by parts

$$\int_{x=1}^{n} xe^{-x}dx = -(x+1)e^{-x}\Big|_{x=1}^{n} = -(n+1)e^{-n} + 2e^{-1} \to 2e^{-1} < \infty$$

as $n \to \infty$.

(4) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n}$.

Answer: The Ratio Test will work to show the first converges and the second diverges. (There is an easier test to use for the second one—do you see it?)

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$.

Answer: Since this is closely related to a geometric series we should use the Comparison Test to compare it to $\sum_{n=1}^{\infty} 1/3^n$ and deduce that it converges.

Chapter 10

Series with Positive and Negative Terms

10.1 Alternating Series

We know that for a series $\sum_{n=1}^{\infty} a_n$ to converge it is definitely *not* sufficient that the sequence $(a_n)_{n\geq 1}$ of terms be null. (It is necessary, but not sufficient.) For example, $1/n \to 0$ as $n \to \infty$, but $\sum_{n=1}^{\infty} 1/n$ does not converge. However, it turns out that if the terms a_n decrease in modulus and alternate in sign:

$$a_1 > 0, \;\; a_2 < 0, \;\; a_3 > 0, \ldots$$

then it is both necessary and sufficient for the convergence of $\sum_{n=1}^{\infty} a_n$ that $(a_n)_{n\geq 1}$ be null.

For example,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

does converge as the following argument suggests:

We have

$$\begin{aligned} 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n+1} - \frac{1}{2n+2} + \cdots &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) + \cdots \\ &= \frac{1}{2} + \frac{1}{12} + \cdots + \frac{1}{(2n+1)(2n+2)} \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)}, \end{aligned}$$

and this series converges by the Comparison Test since

$$\frac{1}{(2n+1)(2n+2)} \leq \frac{1}{(n+1)^2}$$

for all n.

(I say "suggests" here because the argument is not completely rigorous. Why not?) For an argument which gives the sum exactly, see the end of the chapter.

Theorem 10.1.1 (The Alternating Series Test)**.** *Let $(a_n)_{n\geq 1}$ be a decreasing sequence of positive terms such that $a_n \to 0$ as $n \to \infty$. Then the series*

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges.

Proof. Let

$$s_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$$

be a typical partial sum. Then

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} - a_{2n} \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n}. \end{aligned}$$

Notice that all the terms in brackets here are non-negative since the sequence $(a_n)_{n\geq 1}$ is decreasing. Also $a_{2n} > 0$. It follows that for all $n \in \mathbb{N}$, $s_{2n} \leq a_1$.

Also

$$s_{2(n+1)} = s_{2n} + a_{2n+1} - a_{2n+2} \geq s_{2n}$$

since $a_{2n+1} \geq a_{2n+2}$. Hence $(s_{2n})_{n\geq 1}$ is an increasing sequence which is bounded above (by a_1). Therefore it converges by the Monotone Convergence Theorem. Let its limit be ℓ. So $s_{2n} \to \ell$ as $n \to \infty$.

Now consider the sequence $(s_{2n+1})_{n\geq 1}$. Then $s_{2n+1} = s_{2n} + a_{2n+1}$, and both the series s_{2n} and a_{2n+1} converge. So, by the Algebra of Limits Theorem

$$\lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} a_{2n+1} = \ell + 0 = \ell.$$

It now follows easily in this situation that $s_n \to \ell$ as $n \to \infty$, as well, which (by definition) implies that $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges, as required.

In more detail, let $\varepsilon > 0$ be given. Choose N_1 so that $|s_{2n} - \ell| < \varepsilon$ for all $2n \geq N_1$ and N_2 so that $|s_{2n+1} - \ell| < \varepsilon$ for all $2n + 1 \geq N_2$. Then clearly $|s_m - \ell| < \varepsilon$ for all $m \geq N = \max\{N_1, N_2\}$. Hence $s_n \to \ell$ as $n \to \infty$ as required. ■

Remark: Sometimes it is useful to slightly modify the series in the Alternating series. The same argument will work (or we can use the version proved) to deduce either of the following slightly modified versions of the theorem:

(A) *Let $(a_n)_{n\geq 1}$ be a decreasing sequence of positive terms such that $a_n \to 0$ as $n \to \infty$. Then the series*

$$\sum_{n=1}^{\infty} (-1)^n a_n$$

converges.

(B) *Let $(a_n)_{n\geq 0}$ be a decreasing sequence of positive terms such that $a_n \to 0$ as $n \to \infty$. Then the series*

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

converges.

Remark: It might be tempting to say that any alternating sum converges. But that is false: *Let a_n be positive numbers such that $a_n \geq 0$ for all n and (a_n) is decreasing. Then $\sum_{n\geq 1} (-1)^n a_n$ converges* **if and only if** *(a_n) is null.*

Proof. The direction $\Leftarrow$ is The Alternating Series Test 10.1.1. The direction $\Rightarrow$ is the "Nullity Theorem 8.1.4." ■

Example 10.1.2. Let $p \in \mathbb{R}$. Then

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$$

converges if and only if $p > 0$.

Proof: If $p \leq 0$, then for all n,

$$\left| \frac{(-1)^{n+1}}{n^p} \right| = \frac{1}{n^p} \geq 1.$$

So $((-1)^{n+1}/n^p)_{n\geq 1}$ is not a null sequence and hence the series $\sum_{n=1}^{\infty} (-1)^{n+1}/n^p$ cannot converge, by 8.1.4.

On the other hand, if $p > 0$, then $(1/n^p)_{n\geq 1}$ is a decreasing null sequence of positive terms. Hence the series $\sum_{n=1}^{\infty} (-1)^{n+1}/n^p$ converges by the Alternating Series Test. ■

We end this section with a precise formula for the alternating sum $\sum_{n=1}^{\infty}(-1)^{n+1}/n$. The last step in this argument requires the definition of an integral as a limit of sums of areas. If you have not seen this before, do not worry, since it is not something required for this course. But the computation (pointed out by Carolyn Dean) is fun and interesting. This computation is not something you need to remember for this course.

Example 10.1.3. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges to $\ln 2$.

Proof. We know from the Alternating Series Test that the series converges, with sum ℓ, to be found. This means that the sequence of partial sums $(s_n)_{n\geq 1}$ converges to ℓ. This, in turn, means any subsequence converges to ℓ. We will consider $(s_{2n})_{n\geq 1}$.

Claim:

$$s_{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

for all $n \geq 1$.

Proof of the Claim: It is clear that it holds for $n = 1$ since $s_2 = 1 - 1/2 = 1/2$. So suppose that it holds for some $k \geq 1$. Then, by definition

$$\begin{aligned}
s_{2(k+1)} &= 1 - \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2k+1} - \frac{1}{2k+2} \\
&= s_{2k} + \frac{1}{2k+1} - \frac{1}{2k+2} \\
&= \left(\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k}\right) + \frac{1}{2k+1} - \frac{1}{2}\frac{1}{k+1} && \text{by \textit{the inductive hypothesis}} \\
&= \left(\frac{1}{k+2} + \cdots + \frac{1}{2k}\right) + \frac{1}{2k+1} + \frac{1}{2}\frac{1}{k+1} && \text{by \textit{a simple manipulation}.}
\end{aligned}$$

Thus, by induction, the Claim is proven.

Recall from earlier in the course that if $f : [1, +\infty) \to \mathbb{R}$ is a continuous decreasing function then

$$f(n) \geq \int_n^{n+1} f(x)\,dx \geq f(n+1)\,,$$

for all $n \geq 1$. With $f(x) = 1/x$ this can be rearranged as

$$\int_{n-1}^{n} \frac{dx}{x} \geq \frac{1}{n} \geq \int_n^{n+1} \frac{dx}{x}$$

(with $n \geq 2$ for the first inequality). This can be applied immediately to give an upper bound on s_{2n} of

$$s_{2n} \leq \int_n^{2n} \frac{dx}{x} = \ln(2n) - \ln(n) = \ln 2.$$

For a lower bound we first write

$$\begin{aligned} \frac{1}{n+1} + ... + \frac{1}{2n} &= \left(\frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{2n-1}\right) - \frac{1}{n} + \frac{1}{2n} \\ &\geq \int_n^{2n} \frac{dx}{x} - \frac{1}{2n} \\ &= \ln 2 - \frac{1}{2n}. \end{aligned}$$

Combine as

$$\ln 2 \geq s_{2n} \geq \ln 2 - \frac{1}{2n}$$

and let $n \to \infty$ to get $\ell = \ln 2$. ■

10.2 Absolute Convergence

Suppose that we are given a series $\sum_{n=1}^{\infty} a_n$ where some of the a_n are positive and some negative. Then (apart from the Alternating Series Test) there are not so many good rules for deciding whether $\sum_{n=1}^{\infty} a_n$ converges or not. For example something like

$$1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \cdots$$

(where every third term is negative) is not something covered by one of our rules. The one other rule we will have is that the series $\sum_{n=1}^{\infty} a_n$ does converge provided the series $\sum_{n=1}^{\infty} |a_n|$ converges. Before stating the result we make a definition.

Definition 10.2.1. *Let $\sum_{n=1}^{\infty} a_n$ be any series. We say that*

$$\sum_{n=1}^{\infty} a_n$$

is **absolutely convergent** *if the series*

$$\sum_{n=1}^{\infty} |a_n|$$

is convergent.

If $\sum_{n=1}^{\infty} a_n$ is convergent, but not absolutely convergent, then we say it is **conditionally convergent***.*

For example, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is convergent. However, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is conditionally convergent because it converges (see Example 10.1.2) but

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

which diverges.

The definition above rather presupposes that absolute convergence is stronger than convergence and we next give a proof of this fact.

Theorem 10.2.2. *If the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.*

Proof. We are given that $\sum_{n=1}^{\infty} |a_n|$ converges. Let

$$p_n = \begin{cases} a_n & \text{if } a_n \geq 0, \\ 0 & \text{if } a_n \leq 0. \end{cases}$$

Then $\sum_{n=1}^{\infty} p_n$ is a series of positive terms and for all $n \in \mathbb{N}$, $p_n \leq |a_n|$. Hence $\sum_{n=1}^{\infty} p_n$ converges by the Comparison Test. Similarly, if

$$q_n = \begin{cases} |a_n| & \text{if } a_n < 0, \\ 0 & \text{if } a_n \geq 0 \end{cases}$$

then $\sum_{n=1}^{\infty} q_n$ converges.

Hence, by the Algebra of Infinite Sums Theorem 8.1.5, $\sum_{n=1}^{\infty}(p_n - q_n)$ is convergent. But for all $n \in \mathbb{N}$, $p_n - q_n = a_n$ and we are done. ■

Example 10.2.3.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{4}\right)$$

converges absolutely and hence converges.

Proof: The sine terms are

$$\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, , \cdots ,$$

so the alternating series test does not help. However, as $\sin(x) \leq 1$, we do know that

$$0 \leq \left|\frac{1}{n^2} \sin\left(\frac{n\pi}{4}\right)\right| \leq \frac{1}{n^2}$$

for all n. Hence

$$\sum_{n=1}^{\infty} \left|\frac{1}{n^2} \sin\left(\frac{n\pi}{4}\right)\right|$$

converges by the comparison test. ■

The same sort of argument means that we can modify some of our earlier tests to work for series with positive and negative terms. For example:

Theorem 10.2.4. (The Modified Ratio Test) *Suppose that a_n for all $n \in \mathbb{N}$ are any real numbers and assume that*

$$\left|\frac{a_{n+1}}{a_n}\right| \to l$$

as $n \to \infty$.

(i) If $l < 1$, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and hence convergent.

(ii) If $l > 1$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $l = 1$, then we still cannot conclude whether $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.

Proof. (i) In this case the Ratio Test 9.1.7 says that $\sum_{n=1}^{\infty} |a_n|$ converges; i.e. that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(ii) In this case we know from the proof of 9.1.7 that $|a_n| \to \infty$ as $n \to \infty$. So, certainly $a_n \not\to 0$ as $n \to \infty$. Thus, by Theorem 8.1.4, $\sum_{n=1}^{\infty} a_n$ is divergent. ■

Example 10.2.5. $\sum_{n=1}^{\infty} x^n$ (and indeed $\sum_{n=1}^{\infty}(-1)^n x^n$) converges if $|x| < 1$ and diverges if $|x| > 1$. In this case, one can also check that it diverges when $|x| = 1$.

Final Comments. In Example 9.2.10, we saw how one might approach testing series for convergence. Of course, there, the sequences all had positive terms. So how should one approach general series? In a sense the rules are easier, as there are really only three possibilities for a series $\sum_{n=1}^{\infty} a_n$:

1. Is $\lim_{n\to\infty} = 0$? If not then the series diverges by the n-th term test Theorem 8.1.4.
2. Does the Alternating Series Test 10.1.1 apply? If so, use it!
3. Otherwise you had better hope that the Absolute Convergence Test (Theorem 10.2.2) applies, in which case you are back to the ideas of Chapter 9. As we will see in the next chapter, one of the most important cases where this case applies is when one can use the Modified Ratio Test 10.2.4.

For examples of all these cases, see the next Exercise Sheet.

Chapter 11

Power Series

We now consider (the simplest case of) series where the nth term depends on a real variable x and we ask for which values of x does the series converge.

Definition 11.0.1. *A series of the form $\sum_{n=0}^{\infty} a_n x^n$ is called a* **power series** *(in the variable x).*

For example we can write $e^x = \sum_{n=0}^{\infty} x^n/n!$ as a power series. Similarly, the geometric series $\sum_{n=0}^{\infty} x^n$ is a power series (with $a_n = 1$ for all n). As we saw in Example 10.2.5, this converges absolutely for $|x| < 1$ and diverges otherwise. For another example, consider

Example 11.0.2.

$$\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

Let's use the (modified) Ratio Test to study this example; set $c_n = x^n/n$ and apply the rule to $\sum_{n=1}^{\infty} c_n$. So,

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n}\right| = |x| \cdot \frac{n}{n+1} \to |x| \qquad \text{as } n \to \infty.$$

Thus, if $|x| < 1$ the series converges absolutely, by the Modified Ratio Test 10.2.4, while it diverges if $|x| > 1$. When $|x| = 1$ we get two more cases:

If $x = 1$ the series is $\sum_{n=1}^{\infty} 1/n$ which we know diverges.

If $x = -1$ the series is $\sum_{n=1}^{\infty} (-1)^n/n$ which we know converges (by the Alternating Test 10.1.1 and the Remark just after that theorem).

11.1 The Radius of Convergence of a Power Series

You can attempt to find the points at which a power series converges by applying the (modified) ratio test. Given $\sum_{n=0}^{\infty} a_n x^n$ you consider

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x| .$$

If $\lim_{n\to\infty} |a_{n+1}/a_n| = L$, say, exists then $L|x|$ is the ℓ of the ratio test. So, if $L|x| < 1$, i.e. $|x| < 1/L$, the series converges (absolutely) whereas, if $|x| > 1/L$, the series diverges. This suggests that the set of points of converges is an interval centred on the origin. Is this true in all cases or only when you can apply the ratio test?

Another indication that it holds in all cases is

Lemma 11.1.1. *If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x_0 \in \mathbb{R}$, $x_0 \neq 0$, then it converges absolutely for all $y : |y| < |x_0|$.*

Proof. $\sum_{n=0}^{\infty} a_n x_0^n$ converges implies $|a_n x_0^n| \to 0$ as $n \to \infty$. In particular $(|a_n x_0^n|)_{n\geq 1}$ is a convergent sequence. But convergent sequences are bounded so there exists $M > 0$ such that $|a_n x_0^n| < M$ for all $n \geq 1$.

Given $y : |y| < |x_0|$ set $t = |y| / |x_0|$ so $0 \leq t < 1$. Then

$$|a_n y^n| = \left|a_n x_0^n \left(\frac{y}{x_0}\right)^n\right| = |a_n x_0^n| \left|\frac{y}{x_0}\right|^n < Mt^n$$

The geometric series $\sum_{n=0}^{\infty} Mt^n$ converges since $t < 1$ and so, by the Comparison test, $\sum_{n=0}^{\infty} |a_n y^n|$ converges. ■

So again we prove that the series converges in an interval centred on the origin. This is true in general.

Theorem 11.1.2. *Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. then either*

i. the series converges absolutely everywhere or

ii. the series converges only at the origin or

iii. there exists unique $R > 0$ such that the series converges absolutely in $(-R, R)$ and diverges for all $x : |x| > R$.

Proof. Define

$$S = \left\{ |x| : \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\}.$$

This could be written as

$$\left\{ r \geq 0 : \text{either } \sum_{n=0}^{\infty} a_n r^n \text{ or } \sum_{n=0}^{\infty} a_n (-r)^n \text{ converges} \right\}.$$

The set S is non-empty since $0 \in S$. There are two cases, S is unbounded above or S is bounded above.

Assume S is unbounded. Let $x \in \mathbb{R}$ be given. By assumption we can find $r \in S$ with $r > |x|$. But $r \in S$ means $\sum_{n=0}^{\infty} a_n x_r^n$ converges where x_r is one of r or $-r$. Then Lemma says that the power series converges absolutely for all $y : |y| < |x_r| = r$. In particular, at $y = x$. True for all $x \in \mathbb{R}$ means the power series converges absolutely everywhere.

Assume S is bounded. Then by the completeness of $\mathbb{R}$ the set has a least upper bound R, say.

If $x : |x| > R$ then, since R is an upper bound for S we have $|x| \notin S$ and so the power series diverges at a. Hence the series diverges for all $x : |x| > R$.

Note, this holds for all $R \geq 0$. If $R = 0$ then the only possible element of S is 0, at which point the series converges, i.e. the series *only* converges at the origin.

Assume $R > 0$. Assume $x : |x| < R$. If $R \in S$ then Lemma 11.1.1 would immediately give us the convergence at x. But the supremum of a set need not be a member of that set. Instead we have to make use of the fact that R is the lubS.

Since R is the *least* upper bound on S the smaller $|x|$ is **not** an upper bound on S so we can find $r \in S : |x| < r \leq R$. Yet again, $r \in S$ means $\sum_{n=0}^{\infty} a_n x_r^n$ converges where x_r is one of r or $-r$. Then Lemma 11.1.1 says that the power series converges absolutely for all $y : |y| < |x_r| = r$. In particular, at $y = x$. True for all $x : |x| < R$ means the power series converges absolutely in $(-R, R)$. ■

Definition 11.1.3. *If the real number R has the property that the series $\sum_{n=0}^{\infty} a_n x^n$ converges for all x with $|x| < R$, and diverges for all x with $|x| > R$, then R is called the* **radius of convergence** *(***RoC** *for short) of the power series $\sum_{n=0}^{\infty} a_n x^n$.*

We also extend this definition by putting $R = 0$ if $\sum_{n=0}^{\infty} a_n x^n$ only converges for $x = 0$, and putting $R = \infty$ if $\sum_{n=0}^{\infty} a_n x^n$ converges for all $x \in \mathbb{R}$.

By the theorem this definition does cover all possible cases and $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$.

Finding the Radius of Convergence:

To calculate the Radius of Convergence of a given series we always use the Ratio Test. In the course of the calculations in the following examples we tacitly assume that $x \neq 0$. This is justified since a power series always converges for $x = 0$.

The Interval of Convergence.

The set

$$\left\{ x : \sum_{n=0}^{\infty} a_n x^n \text{ converges} \right\}$$

is called the **interval of convergence**. Certainly this contains $(-R, R)$ and it may or may not contain the end points $\pm R$. The Ratio Test will not help at the end points and you need to use some other rule to see if the series converges there.

Example 11.1.4.

$$\sum_{n=0}^{\infty} x^n.$$

- Here we already saw that the Radius of Convergence (RoC) is $R = 1$ and the interval of convergence is $(-1, 1)$.

Example 11.1.5.

$$\sum_{n=1}^{\infty} \frac{1}{n} x^n.$$

- Here we already saw that the RoC is $R = 1$ and the interval of convergence is $[-1, 1)$; that is it converges if $x = -1$ but diverges if $x = +1$.

Example 11.1.6. For

$$\sum_{n=1}^{\infty} n x^n,$$

we have $R = 1$ and it diverges if $x = \pm 1$.

Proof To find the RoC we let $c_n = |nx^n|$. Then $\sum_{n=1}^{\infty} c_n$ is a series of positive terms and

$$\frac{c_{n+1}}{c_n} = \frac{(n+1)|x|^{n+1}}{n|x|^n} = \left(1 + \frac{1}{n}\right) \cdot |x| \to (1+0) \cdot |x| = |x|, \text{ as } n \to \infty.$$

So the (Modified) Ratio Test now tells us that if $|x| < 1$ then $\sum_{n=1}^{\infty} c_n$ converges absolutely, whereas if $|x| > 1$ then $\sum_{n=1}^{\infty} c_n$ diverges. This shows that the RoC is 1. It is left for you to check that it diverges if $x = \pm 1$. ■

Example 11.1.7. For

$$\sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3} x^n,$$

we have $R = 1/27$.

Proof Let $c_n = |(3n)!x^n/(n!)^3|$. Then $c_n > 0$ and

$$\begin{aligned}
\frac{c_{n+1}}{c_n} &= \frac{(3(n+1))! \cdot |x|^{n+1}}{((n+1)!)^3} \cdot \frac{(n!)^3}{(3n)! \cdot |x|^n} \\
&= \frac{(3n)!(3n+1)(3n+2)(3n+3) \cdot |x|^{n+1}}{(n!)^3(n+1)^3} \cdot \frac{(n!)^3}{(3n)! \cdot |x|^n} \\
&= \frac{(3n+1)(3n+2)(3n+3)}{(n+1)(n+1)(n+1)} \cdot |x| \\
&= \frac{(3+\frac{1}{n})(3+\frac{2}{n})(3+\frac{3}{n})}{(1+\frac{1}{n})(1+\frac{1}{n})(1+\frac{1}{n})} \cdot |x| \\
&\to \frac{(3+0)(3+0)(3+0)}{(1+0)(1+0)(1+0)} \cdot |x| \quad (\text{as } n \to \infty) \\
&= 27|x|.
\end{aligned}$$

So by the Ratio Test, $\sum_{n=1}^{\infty} c_n$ converges for $27|x| < 1$, i.e. for $|x| < 1/27$, and diverges for $|x| > 1/27$. Therefore $\sum_{n=1}^{\infty} |(3n)!x^n/(n!)^3|$ converges for $|x| < 1/27$ and diverges for $|x| > 1/27$. Thus, as in the previous example (and, indeed, all examples like this), we conclude that the RoC of the series $\sum_{n=1}^{\infty}(3n)!x^n/(n!)^3$ is $1/27$. ■

Example 11.1.8. For

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

we have $R = \infty$,

Proof Let $c_n = |x^n/n!|$ and, just as in Example 9.1.9, we see that $c_{n+1}/c_n \to 0$ as $n \to \infty$ no matter what x is. Since $0 < 1$ it follows that the radius of convergence of this series (the exponential series) is ∞. ■

Example 11.1.9. For $\sum_{n=0}^{\infty} n!\, x^n$ we have $R = 0$.

Proof Let $c_n = |n!\, x^n|$, then

$$\frac{c_{n+1}}{c_n} = \frac{(n+1)!\, x^{n+1}}{n!\, x^n} = (n+1)x.$$

Notice that here *for any* $x \neq 0$, $(n+1)x \to \infty > 1$ as $n \to \infty$. Thus the Ratio Test says that the series diverges for any $x \neq 0$. Thus the RoC of the series $\sum_{n=0}^{\infty} n!\, x^n$ is 0: the series converges for no value of x except $x = 0$.

11.2 The n-th Root Test

Theorem 11.2.1. ***The*** n***-th root test*** *Starting with a series* $\sum_{n=1}^{\infty} a_n$ *assume that the limit* $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \ell$ *exists. Then*

i. if $\ell < 1$*, the series converges absolutely,*

ii. if $\ell > 1$*, the series diverges,*

iii. if $\ell = 1$ *the test tells us nothing.*

Proof. Assume $\ell < 1$. Choose $\varepsilon : \ell + \varepsilon = (1+\ell)/2$, i.e. $\varepsilon = (1-\ell)/2 > 0$, in the definition of limit to find $N_1 \geq 1$ such that if $n \geq N_1$ then $\left|\sqrt[n]{|a_n|} - \ell\right| < \varepsilon$. In particular, $\sqrt[n]{|a_n|} - \ell < \varepsilon$, i.e.

$$\sqrt[n]{|a_n|} < \ell + \varepsilon = \frac{1+\ell}{2} = L,$$

say. Because $\ell < 1$ we have $L < 1$. And $|a_n| < L^n$ for all $n \geq N_1$. The geometric series $\sum_{n=N_1}^{\infty} L^n$ converges since $L < 1$ and so, by the Comparison Test, $\sum_{n=N_1}^{\infty} |a_n|$ converges, as does then $\sum_{n=1}^{\infty} |a_n|$. Hence $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Assume $\ell > 1$. Choose $\varepsilon : \ell - \varepsilon = (1+\ell)/2$, i.e. $\varepsilon = (\ell - 1)/2 > 0$, in the definition of limit to find $N_2 \geq 1$ such that if $n \geq N_2$ then $\left|\sqrt[n]{|a_n|} - \ell\right| < \varepsilon$. In particular, $-\varepsilon < \sqrt[n]{|a_n|} - \ell$, i.e.

$$\sqrt[n]{|a_n|} > \ell - \varepsilon - \frac{1+\ell}{2} = L,$$

though this time $L > 1$ since $\ell > 1$. then $|a_n| > L^n > 1$ for all $n \geq N_2$. This means the series diverges since the terms do not tend to zero as $n \to \infty$. ■

Example 11.2.2. Determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \left(\frac{2n^3 - 5}{(3n^2+1)(n+5)} \right)^n .$$

Proof.

$$\sqrt[n]{|a_n|} = \left| \frac{2n^3 - 5}{(3n^2+1)(n+5)} \right| = \left| \frac{2 - 5/n^3}{(3 + 1/n^2)(1 + 5/n)} \right| \to \frac{2}{3}$$

as $n \to \infty$. Since $2/3 < 1$ the series converges absolutely. ■

Example 11.2.3. Determine if $\sum_{n=1}^{\infty} a_n$ is convergent or divergent when

$$a_{2n+1} = \left(\frac{1}{3}\right)^n 2^n \quad \text{and} \quad a_{2n} = \left(\frac{1}{3}\right)^{n+1} 2^n.$$

for all as $n \geq 1$. So the series starts

$$1, \frac{2}{9}, \frac{2}{3}, \frac{4}{27}, \frac{4}{9}, \ldots$$

Why have I asked this question?

Proof.

$$\sqrt[n]{|a_{2n+1}|} = \frac{2}{3} \quad \text{and} \quad \sqrt[n]{|a_{2n}|} = \frac{2}{3^{1+1/n}} \to \frac{2}{3}$$

as $n \to \infty$. Thus $\sqrt[n]{|a_n|} \to 2/3$ as $n \to \infty$. Since $2/3 < 1$ the series converges.

I have asked this question because the ratio test will not work in this example; try it.

■

Example 11.2.4. Determine the interval of convergence of

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n x^n.$$

Proof.

$$\left|\left(1 + \frac{1}{n}\right)^n x^n\right|^{1/n} = \left|\left(1 + \frac{1}{n}\right)\right| |x| \to |x|$$

as $n \to \infty$. Thus, by the n-th root test, the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. Thus the radius of convergence is 1.

At the end points, 1 and -1 we note that, by the first two terms from the Binomial Theorem,

$$\left(1 + \frac{1}{n}\right)^n \geq 1 + \frac{n}{n} = 2$$

for all $n \geq 1$, and so the terms of the series do not converge to 0 and thus the series diverges. Hence the interval of convergence is $(-1, 1)$. ■

Example 11.2.5. What is $\lim_{n\to\infty} n^{1/\sqrt{n}}$? If $N \geq 1$ is an integer and we set $k = n^{1/N} - 1$ perhaps you can look back at the proof of $\lim_{n\to\infty} n^{1/n} = 1$ and prove

$$k < \left(3!\frac{n}{N(N-1)(N-2)}\right)^{1/3}.$$

Choosing N to be the nearest integer to $\sqrt{n}$ might give the answer to the original question.

Example 11.2.6. Determine the radius of convergence of

$$\sum_{n=1}^{\infty} \frac{x^n}{n^{\sqrt{n}}}.$$

Proof.

$$\left|\frac{x^n}{n^{\sqrt{n}}}\right|^{1/n} = \frac{|x|}{n^{1/\sqrt{n}}} \to |x|$$

as $n \to \infty$. Thus the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. Hence the radius of convergence is 1. It also converges at both ends of interval of convergence so the interval of convergence is $[-1, 1]$.

■

Chapter 12

Further Results on Power Series

12.1 More General Taylor Series.

First, we have only been discussing power series of the form $\sum_{n=0}^{\infty} a_n x^n$. However when working with Taylor series one often wants to work with expansions around values α other than zero, in which case one would get an expression like $\sum_{n=0}^{\infty} a_n(x-\alpha)^n$. Using a substitution $y = (x - \alpha)$ allows one to reduce to the case we have been considering and so essentially all the same theorems will hold. For example, one gets the following variant of Theorem 11.1.2:

Theorem 12.1.1. *Consider the series $\sum_{n=0}^{\infty} a_n(x-\alpha)^n$, for some fixed number α. Then there are three possibilities concerning convergence.*

Case (i): $\sum_{n=0}^{\infty} a_n(x-\alpha)^n$ converges only for $x = \alpha$.

Case (ii): $\sum_{n=0}^{\infty} a_n(x-\alpha)^n$ converges for all $x \in \mathbb{R}$.

Case(iii): There exists a unique positive real number R such that

(a) $\sum_{n=0}^{\infty} a_n(x-\alpha)^n$ converges absolutely for all x with $|x-\alpha| < R$, and

(b) $\sum_{n=0}^{\infty} a_n(x-\alpha)^n$ diverges for all x with $|x-\alpha| > R$.

Proof. Set $y = x-\alpha$. Then our series becomes $\sum_{n=0}^{\infty} a_n y^n$. So, now apply Theorem 11.1.2 to that series and you will find the conclusion you get is exactly the present theorem. For example, if $\sum_{n=0}^{\infty} a_n y^n$ has radius of convergence R then $\sum_{n=0}^{\infty} a_n y^n$ converges if $|y| < R$, which is the same as saying that

$$\sum_{n=0}^{\infty} a_n(x-\alpha)^n = \sum_{n=0}^{\infty} a_n y^n$$

converges if $|x-\alpha| < R$. Similarly it diverges if $|x-\alpha| > R$. ■

12.2 Rearranging Series

Let us begin with the following remarkable example. Given a series $\sum_{n=1}^{\infty} a_n$, then a **rearrangement** of this series means a series of the form $\sum_{n=1}^{\infty} b_n$ which is got by rearranging the terms of the first one. More formally, there exists a bijection $\phi : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{\phi(n)}$ for each n.

Here is a famous example.

Example 12.2.1. Recall from Example 10.1.3 that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2).$$

Here is a rearrangement with sum $\ln(2)/2$. To do this we use the rearrangement where we always have two negative terms in succession but only one positive one:

$$\begin{aligned} &1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \cdots \\ &= \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) \cdots \\ &= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{10} - \frac{1}{12}\right) \cdots \\ &= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots\right) = \frac{\ln(2)}{2}. \end{aligned}$$

In fact we can get any number we want in this way.

Proposition 12.2.2. *Suppose that $\sum_{n\geq 1} a_n$ is a conditionally convergent series, and let α be* **any** *real number. Then there exists some rearrangement $\sum_{n\geq 1} b_n$ of this series for which $\sum_{n\geq 1} b_n = \alpha$.*

Proof. (Outline) We start with some notation essentially coming from Question 3 on the Exercise sheet for Week 11:

Notation 12.2.3. Given a series $\sum_{n\geq 1} a_n$, write

$$a_n^+ = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n \leq 0 \end{cases} \qquad \text{and} \qquad a_n^- = \begin{cases} 0 & \text{if } a_n \geq 0 \\ a_n & \text{if } a_n \leq 0 \end{cases}$$

By that exercise sheet, the series a_n^+ and a_n^- are divergent. Hence $\sum a_n^+ = \infty$, since all the terms are positive and, similarly, $\sum a_n^- = -\infty$.

Now, we construct the sequence $\{b_n\}$: Since $\sum a_n^+ = \infty$ we can therefore take the first few positive numbers $a_{k_1}, a_{k_2} \dots, a_{k_r}$ so that $X_1 = a_{k_1} + a_{k_2} + \cdots + a_{k_r} > \alpha$. (More precisely we choose the smallest possible r with this property.) Then, as $\sum a_n^- = -\infty$, we can add to this the first few negative numbers $a_{k_{r+1}}, a_{k_{r+2}} \dots, a_{k_s}$ so that

$$X_2 = (a_{k_1} + a_{k_2} + \cdots + a_{k_r}) + \big(a_{k_{r+1}} + a_{k_{r+2}} + \cdots + a_{k_s}\big) < \alpha.$$

(Again we take s minimal with this property.) Now keep going; adding positive terms a_j to get bigger than α then immediately adding more negative ones so that the sum becomes less than α, then immediately adding more positive terms *et cetera*.

Finally, we must check that the sum actually converges to α. This is the same idea as in the proof of the Alternating Series test, but the notation is messy. More formally, we have constructed the numbers $X_1 > \alpha$, then $X_2 < \alpha$, and $X_3 > \alpha$, etc. Let a_{ℓ_u} be the last number added to make X_u; thus in the last paragraph $a_{\ell_1} = a_{k_r}$ while $a_{\ell_2} = a_{k_s}$. The key point to notice is that, as each a_{ℓ_j} is chosen minimally, $|X_j - \alpha| \leq |a_{\ell_j}|$ in each case. But, by the n^{th} term test Theorem 8.1.4, $\lim_{j\to\infty} |a_{\ell_j}| \to 0$ as $j \to \infty$. Therefore, by the Sandwich Theorem 3.1.4, $|X_j - \alpha| \to 0$ as $j \to \infty$. Which is exactly what we wanted to prove. ■

Strangely enough, for *absolutely convergent* series, taking rearrangements does not change the sum.

Theorem 12.2.4. *Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series, say with $\sum_{n=1}^{\infty} a_n = \ell$. Then for any rearrangement $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ we have $\sum_{n=1}^{\infty} b_n = \ell$, as well.*

Proof. Define the a_n^+, a_n^- and similarly b_n^+, b_n^- as in Notation 12.2.3. By Question 3(a) of the Exercise sheet for Week 11, $\sum_{n=1}^{\infty} a_n^+$ is convergent, say with $A = \sum_{n=1}^{\infty} a_n^+$. Also, obviously the partial sums $\{u_\ell = \sum_{n=1}^{\ell} a_n^+\}$ form an increasing sequence. Now, for any t, the terms $b_1^+, \dots, b_t^+$ appear in $\{a_1^+, \dots, a_\ell^+\}$ for some ℓ and hence the partial sums

$$v_t = \sum_{n=1}^{t} b_n^+ \leq \sum_{n=1}^{\ell} a_n^+ \leq \sum_{n=1}^{\infty} a_n^+.$$

Hence $\{v_t\}_{t\geq 1}$ is an ascending and bounded sequence. Thus, by the Monotone Convergence Theorem, it has a limit, say B. Notice also that $B \leq A$ by Lemma 4.2.5. In particular this also shows that $\sum b_n$ is absolutely convergent (if not then Question 3 of the Exercise sheet for Week 11 would imply that $\sum b_n^+ = \infty$). So we can reverse this argument and see that each $u_\ell \leq B$ and hence that $A \leq B$.

In other words $A = B$.

Now repeat this argument for the a_n^- (with decreasing sequences of partial sums) to show that $\sum_{n\geq 1} a_n^- = \sum_{n\geq 1} b_n^-$. Now $\sum_{n\geq 1} a_n = \sum_{n\geq 1} a_n^+ + \sum_{n\geq 1} a_n^-$ (use Question 3(a) of the Exercise sheet for Week 11, again), and similarly $\sum_{n\geq 1} b_n = \sum_{n\geq 1} b_n^+ + \sum_{n\geq 1} b_n^-$. Therefore we conclude that

$$\sum_{n\geq 1} a_n \;=\; \sum_{n\geq 1} a_n^+ + \sum_{n\geq 1} a_n^- \;=\; \sum_{n\geq 1} b_n^+ + \sum_{n\geq 1} b_n^- \;=\; \sum_{n\geq 1} b_n.$$

■

One significant application of this result is that it tells us what happens when we multiply power series. To set this up, recall the equation $e^x \cdot e^y = e^{x+y}$. In terms of power series this should mean that

$$\left(\sum_{n\geq 0} \frac{x^n}{n!}\right)\left(\sum_{n\geq 0} \frac{y^n}{n!}\right) = \left(\sum_{n\geq 0} \frac{(x+y)^n}{n!}\right).$$

("should" because when you construct exponentials formally in next year's Real Analysis course you will proceed in the opposite direction: you *define* the exponential e^x as the power series $\sum_{n=0}^{\infty} x^n/n!$. This for example solves the problem of what exactly e is, but it does mean that rules like $e^x e^y = e^{x+y}$ are no longer "obvious".)

Anyway, this displayed equation is true and actually works much more generally. To prove this, we start with an analogous result for series. Here it is more natural to start summing our power series at $n = 0$ rather than $n = 1$ and so the next few results will be phrased in that way.

Theorem 12.2.5. *Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be absolutely convergent series, say with $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$.*

Write $\sum_{i,j\geq 0} a_i b_j$ for the infinite sum consisting of the product, in any order, of every term of the first series multiplied by every term of the second series. Then $\sum_{i,j\geq 0} a_i b_j$ is absolutely convergent (in the sense that the sum $\sum_{i,j\geq 1} |a_i b_j|$ is also convergent) and $\sum_{i,j\geq 0} a_i b_j = AB$.

Remark: One use of this theorem is when we need to make sense of a summation with two indices in an expression like $\sum_{i,j\geq 0} a_i b_j$. The problem is that one does not want to take something like $\sum_{j=0}^{\infty}(\sum_{i=0}^{\infty} a_i b_j)$, since now one has an infinite sum of infinite sums—which causes a whole new set of problems! We get around this by making it into a single infinite sum $\sum_{m\geq 0} c_m$. But then one runs into the problem that there are many different ways of choosing the sum. The theorem says this does not matter since whatever way you do it you get the same answer.

Proof. Recall from the Foundations of Pure Mathematics course that $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ is countable, so pick your favourite bijection $\phi : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, and write it as $\phi(m) = (\phi_1(m), \phi_2(m))$. Now set $c_m = a_{\phi_1(m)} b_{\phi_2(m)}$; thus the definition of $\sum_{i,j\geq 0} a_i b_j$ in the statement of the theorem just means $\sum_{m\geq 0} c_m$. Then, by considering the partial sums as in the previous theorem, we see that the partial sums $\gamma_t = \sum_{n=0}^{t} |c_m|$ form an increasing sequence of non-negative terms and certainly

$$\gamma_t \ \leq \ X = \left(\sum_{n=0}^{\infty} |a_n|\right)\left(\sum_{n=0}^{\infty} |b_n|\right) \quad \text{for} \quad t \in \mathbb{N}.$$

Therefore, by the Monotone Convergence Theorem again, the sequence $\{\gamma_t\}$ has a limit bounded above by X. In other words

$$\gamma = \sum_{m=0}^{\infty} |c_m| \ \leq \ X \ = \ \left(\sum_{n=0}^{\infty} |a_n|\right)\left(\sum_{n=0}^{\infty} |b_n|\right).$$

But we can now apply Theorem 12.2.4 to conclude that, however we ordered the terms, $\sum_{i,j\geq 1} a_i b_j$ must give the same number γ. Now, one way of constructing the doubly infinite sum is for the $(N+1)^2$ term to be $\left(\sum_{n=0}^{N} a_n\right)\left(\sum_{n=0}^{N} b_n\right)$. Since

$$\left(\sum_{n=0}^{N} a_n\right)\left(\sum_{n=0}^{N} b_n\right) \to AB \qquad \text{as } N \to \infty,$$

it follows that in fact our sum $\sum_{i,j\geq 0} a_i b_j$ equals AB, as well.

Notice that since we only get one possible sum, Proposition 12.2.2 says the series must be absolutely convergent. ■

As a special case of the theorem we get:

Corollary 12.2.6. (Cauchy Product) *Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be absolutely convergent series, say with $\sum_{n=0}^{\infty} a_n = A$ and $\sum_{n=0}^{\infty} b_n = B$. Set $c_n = \sum_{r=0}^{n} a_r b_{n-r}$ for each n.*

Then $\sum_{n=0}^{\infty} c_n$ converges absolutely with $\sum_{n=0}^{\infty} c_n = AB$.

Proof. Once again, the given sequence of partial sums $\sum_{n=0}^{t} c_n$ is a subsequence of some sequence of partial sums constructed from $\sum_{i,j\geq 0} a_i b_j$. Therefore, by Theorem 6.1.3, it also converges to AB. Once again, as we only get one possible sum, Proposition 12.2.2 says the series must be absolutely convergent. ■

Finally we get the result on exponentials that we wanted:

Corollary 12.2.7. *Let*

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Then $E(x)$ is absolutely convergent for all x. Moreover for any $x, y \in \mathbb{R}$ one has $E(x)E(y) = E(x+y)$.

Proof. Of course $E(x)$ is absolutely convergent for all x, by Example 11.1.8. The second sentence follows from the Corollary 12.2.6 once one notices that (by using the Binomial Theorem)

$$\frac{(x+y)^n}{n!} = \frac{1}{n!}\sum_{r=0}^{n} \frac{n!}{r!(n-r)!} x^r y^{n-r} = \sum_{r=0}^{n} \left(\frac{x^r}{r!}\right)\left(\frac{y^{n-r}}{(n-r)!}\right).$$

■

As you might imagine, the Cauchy Product Theorem 12.2.6 and its variants can be used to prove lots of other product formulas.

12.3 Analytic Functions

This section gives an indication of where you will go with power series and related topics in future courses.

Let R be a positive real number or ∞ and suppose that the series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R. Then for each x in the interval $(-R, R)$ the series $\sum_{n=0}^{\infty} a_n x^n$ converges. Let us call its sum $f(x)$. Then $f : (-R, R) \to \mathbb{R}$ and

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Functions that can be obtained this way are called *analytic* (on $(-R, R)$).

One can now show that this analytic function f can be differentiated and that $f'(x)$

is given by the result of differentiating the series term by term:

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}.$$

A fundamental fact is that the differentiated series has the same radius of convergence as the original series, namely R. So we may differentiate again:

$$f''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

and repeat this for any $k \in \mathbb{N}$ to obtain

$$f^{(k)}(x) = \sum_{n=0}^{\infty} n(n-1)(n-2)\cdots(n-k+1) a_n x^{n-k}.$$

Now the negative exponents of x do not actually appear (as their coefficients contain a 0, so vanish) and we usually rewrite this series (by changing the variable of summation from n to $n+k$) as

$$f^{(k)}(x) = \sum_{n=0}^{\infty} a_{n+k}(n+k)(n+k-1)(n+k-2)\cdots(n+1)x^n.$$

Now if we put $x = 0$ in this expression then all the terms except the first vanish, and we obtain the following formula for the coefficients of the original series in terms of the function f:

$$f^{(k)}(0) = k! a_k$$

or

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

So, to summarise, analytic functions behave very nicely with respect to differentiation (and integration and almost all other operations) and the coefficients a_n can easily be determined. These functions therefore are very convenient to work with.

In the other direction suppose now that, rather than a power series, we are given a function $f : (-R, R) \to \mathbb{R}$ which can be differentiated as many times as we please. Does it follow that it is an analytic function? Certainly we know what its power series must be, by the formula above, namely

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

This power series is called the *Taylor series* of the function f and one might guess that it converges to $f(x)$ for all $x \in (-R, R)$. If you have calculated examples of Taylor series before, like for e^x, $\sin x$ or $\ln(1+x)$, then this has always been the case. And, indeed, it is true with any "reasonable" function.

However, one can easily write down functions for which the Taylor series does not converge (except at $x = 0$) and, rather more shockingly, there are also examples of functions $f : \mathbb{R} \to \mathbb{R}$ whose Taylor series converges for all x (i.e. the radius of convergence is ∞) but do not converge to $f(x)$ ...