1 Counting Collections of Functions and of Subsets.

See p.144.

All page references are to P.J.Eccles book unless otherwise stated.

Let X and Y be sets.

Definition 1.1 *$Fun\,(X,Y)$ will be the set of **all** functions from X to Y and $Inj\,(X,Y)$ will be the set of all **injections** from X to Y.*

> **Advice for exams**: I consider that when, in an exam paper, I ask you to give a definition then that is the opportunity for you to gain easy marks.
>
> **Why should you learn all the definitions?** (Other than doing well in the exams?) How can you check that a map between sets is a surjection if you don't know what a surjective function is? If you are told a function is injective how can you use that information if you don't know the definition of injective?

Example 1.2 *Let $X = \{a, b\}$ and $Y = \{1, 2, 3\}$.*

$$\begin{array}{lll} f_1 : \left\{ \begin{array}{l} a \mapsto 1 \\ b \mapsto 1 \end{array} \right., & f_2 : \left\{ \begin{array}{l} a \mapsto 1 \\ b \mapsto 2 \end{array} \right., & f_3 : \left\{ \begin{array}{l} a \mapsto 1 \\ b \mapsto 3 \end{array} \right., \\ f_4 : \left\{ \begin{array}{l} a \mapsto 2 \\ b \mapsto 1 \end{array} \right., & f_5 : \left\{ \begin{array}{l} a \mapsto 2 \\ b \mapsto 2 \end{array} \right., & f_6 : \left\{ \begin{array}{l} a \mapsto 2 \\ b \mapsto 3 \end{array} \right., \\ f_7 : \left\{ \begin{array}{l} a \mapsto 3 \\ b \mapsto 1 \end{array} \right., & f_8 : \left\{ \begin{array}{l} a \mapsto 3 \\ b \mapsto 2 \end{array} \right., & f_9 : \left\{ \begin{array}{l} a \mapsto 3 \\ b \mapsto 3 \end{array} \right.. \end{array}$$

Then

$$Fun\,(X,Y) = \{f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9\}\,,$$

and

$$Inj\,(X,Y) = \{f_2, f_3, f_4, f_6, f_7, f_8\}\,.$$

Theorem 1.3 *Assume* X *and* Y *are finite sets and set* $m = |X|$ *and* $n = |Y|$. *Then*

$$|Fun(X,Y)| = n^m$$

and

$$|Inj(X,Y)| = n(n-1)...(n-m+1) = \frac{n!}{(n-m)!},$$

for $n \geq m$ *with the convention that* $0! = 1$.

Proof pp 145, 146. But here I give a *non-rigorous* proof, in terms of the number of *choices* of functions.

Question In how many ways can we select a function from X to Y?

Answer A function is defined by the images of each element in X. List the elements in X. How many choices for an image for the first element of X? There are n such choices. For the second element of X. Again n choices. Third element? n choices and in fact n choices for every element of X. We multiply choices together to get a total of $n \times n \times n \times ... \times n$, m times, or n^m choices of functions.

For injective functions there are n choices for the images of the first element of X. But with an element of Y now "used" there are only $n-1$ choices for the image of the second element of Y. This "uses up" two elements of Y and so there are only $n-2$ choices for the third element of X. Continue and we find a total of

$$n \times (n-1) \times (n-2) \times (n-m+1) = \frac{n!}{(n-m)!}$$

choices of injective functions. ■

Advice for the exams. This is the first of many theorems in this course. One of the aims of this course is to get you aquainted with proofs and mathematical reasoning. In fact the title of Peter Eccle's book, the major reference for this course, is Introduction to Mathematical Reasoning, not Sets, Numbers and Functions.

It is too much to expect undergraduates to find their own proofs of results so the only way we can judge that you understand mathematical proofs and logical reasoning is to get you to learn and write out proofs in exams.

Perhaps it is too daunting to learn all the proofs so

• start learning them immediately, do not leave revision until the last minute. Unfortunately you won't remember a proof by reading it, you will have to write it out (probably a number of times).

• note that however long a proof it normally contains only one 'idea'. Remember that idea and the rest of the proof often follows.

You can start, though, with learning the statements of the proofs. You should attempt to memorise them so well that you can write them down with no thought. As with definitions if I ask for the statement of a Theorem in the exam then that is the opportunity for you to gain easy marks. My recommendation for how to learn these proofs is to write them out - many times. But then the syllabus for this course says that you should put aside 145 independent study hours!

Example 1.4 *In the example above of $X = \{a, b\}$ and $Y = \{1, 2, 3\}$ we have*

$$|Fun\,(X, Y)| = 9 = 3^2$$

and

$$|Inj\,(X, Y)| = 6 = \frac{3!}{1!}.$$

A special case is, for any **finite** set A,

Corollary 1.5 *If A is finite then*

$$|Fun\,(A, \{0, 1\})| = 2^{|A|}.$$

Recall

Definition 1.6 *For a set A the* ***Power set*** *$\mathcal{P}(A)$ is the collection of* ***all*** *subsets of A, so*

$$\mathcal{P}(A) = \{C : C \subseteq A\}.$$

We will give an alternative way to calculate the cardinality of $\mathcal{P}$ for which we will need

Definition 1.7 *Given a set U and subset $C \subseteq U$ define the* ***characteristic function*** *of C by $\chi_C : U \to \{0, 1\}$,*

$$\chi_C(a) = \begin{cases} 1 & \text{if } a \in C \\ 0 & \text{otherwise.} \end{cases}$$

So the collection of all characteristic functions on a set A is $Fun\left(A,\{0,1\}\right)$. By finding a bijection between $\mathcal{P}\left(A\right)$ and $Fun\left(A,\{0,1\}\right)$ we can prove

Theorem 1.8 *Let A be a finite set. Then*

$$|\mathcal{P}\left(A\right)| = 2^{|A|}.$$

Proof Define a map from *sets* to *functions* by

$$\begin{aligned} \mathcal{P}\left(A\right) &\rightarrow Fun\left(A,\{0,1\}\right), \\ C &\mapsto \chi_C. \end{aligned}$$

This map has an inverse

$$\begin{aligned} Fun\left(A,\{0,1\}\right) &\rightarrow \mathcal{P}\left(A\right), \\ f &\mapsto C_f = \{a \in A : f\left(a\right) = 1\}. \end{aligned}$$

We do not show here that the maps are inverses of each other. For that, see Appendix 1.

A function with an inverse is a bijection, thus we have a bijection between $\mathcal{P}\left(A\right)$ and $Fun\left(A,\{0,1\}\right)$, and so the sets have the same cardinality. That is,

$$\begin{aligned} |\mathcal{P}\left(A\right)| &= |Fun\left(A,\{0,1\}\right)| \\ &= 2^{|A|}, \end{aligned}$$

by the result above. So we have again shown that

$$|\mathcal{P}\left(A\right)| = 2^{|A|}.$$

See p.148 ■

Definition 1.9 *For a set A the set $\mathcal{P}_r\left(A\right)$ is the collection of all subsets of A containing exactly r elements. So*

$$\mathcal{P}_r\left(A\right) = \{C \subseteq A : |C| = r\},$$

or, equivalently,

$$\mathcal{P}_r\left(A\right) = \{C \in \mathcal{P}\left(A\right) : |C| = r\}.$$

Example 1.10 *If $A = \{a, b, c, d, e\}$ then*

$$\begin{aligned}\mathcal{P}_3(A) &= \{\ \{a,b,c\},\ \{a,b,d\},\ \{a,b,e\},\ \{a,c,d\},\ \{a,c,e\} \\ &\qquad \{a,d,e\},\ \{b,c,d\},\ \{b,c,e\},\ \{b,d,e\},\ \{c,d,e\}\ \}.\end{aligned}$$

Also, $\mathcal{P}_0(A) = \{\varnothing\}$ and $\mathcal{P}_5(A) = \{A\}$.

We now restrict to **finite sets** A.

Definition 1.11 *The **binomial number** $\binom{n}{r}$, alternatively $C(n,r)$ or ${}^nC_r$, read as "n choose r" is the cardinality $|\mathcal{P}_r(A)|$, for any set A of cardinality n, i.e.*

$$\binom{n}{r} = |\mathcal{P}_r(A)|.$$

Example 1.12 *With $A = \{a, b, c, d, e\}$ then from above we see that $|\mathcal{P}_3(A)| = 10$ so*

$$\binom{5}{3} = 10.$$

Also from above,

$$\begin{aligned}|\mathcal{P}_0(A)| &= 1, \quad \text{i.e. } \binom{5}{0} = 1 \\ |\mathcal{P}_5(A)| &= 1, \quad \text{i.e. } \binom{5}{5} = 1.\end{aligned}$$

In fact, if $|A| = n$ then

$$\mathcal{P}_0(A) = \{\varnothing\} \quad \text{and} \quad \mathcal{P}_n(A) = \{A\},$$

in which case

$$\binom{n}{0} = 1 \ \text{ and } \ \binom{n}{n} = 1 \text{ for all } n \geq 1.$$

And further, if $A = \varnothing$, it has one and only one subset, namely $\varnothing$ and so

$$\binom{0}{0} = 1.$$

Question Does this definition of binomial number depend on the choice of set A?

Theorem 1.13 *If A and B both contain n elements then, for all $r \geq 1$,*

$$|\mathcal{P}_r(A)| = |\mathcal{P}_r(B)|.$$

Proof Not given in lectures (and so not examinable) but **see** p.149 and p.110.

The idea starts from A and B both containing n elements and so they have the same cardinality. Thus there exists a bijection $f : A \to B$. As on p. 110 extend this to a function between the power sets

$$\begin{aligned} \overrightarrow{f} : \mathcal{P}(A) &\to \mathcal{P}(B) \\ C &\mapsto \overrightarrow{f}(C) = \{f(c) : c \in C\}. \end{aligned}$$

So the image of C under $\overrightarrow{f}$ is the set of all images of the elements of C under f.

The extended function $\overrightarrow{f}$ has an inverse function given by

$$\begin{aligned} \overleftarrow{f} : \mathcal{P}(B) &\to \mathcal{P}(A) \\ D &\mapsto \overleftarrow{f}(D) = \left\{f^{-1}(d) : d \in D\right\}, \end{aligned}$$

So the image of D under $\overleftarrow{f}$ is the set of all pre-images of the elements of D under f^{-1}.

We do not show here that $\overrightarrow{f}$ and $\overleftarrow{f}$ are inverse, instead see the appendix for the details. But being inverses implies that $\overrightarrow{f} : \mathcal{P}(A) \to \mathcal{P}(B)$ is a bijection in which case $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

We can go further, since f is a bijection it is an injection and thus the cardinality of a set C and its image, $\overrightarrow{f}(C)$, are equal. That is

$$\left|\overrightarrow{f}(C)\right| = |\{f(c) : c \in C\}| = |C|,$$

i.e. we "lose" no elements when we look at the images $f(c)$ of elements $c \in C$. Thus if $C \in \mathcal{P}_r(A)$, and thus of cardinality r, then $\overrightarrow{f}(C)$ is *also* of cardinality r, i.e. $\overrightarrow{f}(C) \in \mathcal{P}_r(B)$. Therefore we have a bijection

$$\overrightarrow{f} : \mathcal{P}_r(A) \to \mathcal{P}_r(B),$$

in which case $|\mathcal{P}_r(A)| = |\mathcal{P}_r(B)|$. ■

To help us calculate $|\mathcal{P}_r(A)|$ we use an inductive method, relating $|\mathcal{P}_r(A)|$ to $|\mathcal{P}_{r'}(A')|$ with either a smaller $r' < r$ or proper subset $A' \subset A$.

Theorem 1.14 *Let A be a set containing $n \geq 1$ elements and let $a \in A$ be any such element. Then, for $1 \leq r \leq n$,*

$$|\mathcal{P}_r(A)| = |\mathcal{P}_{r-1}(A \setminus \{a\})| + |\mathcal{P}_r(A \setminus \{a\})|.$$

Proof p.150 We will construct a bijection to show that

$$|\mathcal{P}_r(A)| = |\mathcal{P}_{r-1}(A \setminus \{a\}) \cup \mathcal{P}_r(A \setminus \{a\})|.$$

Then since the union is a *disjoint* union (there are no sets of size r and $r-1$ simultaneously) we get the stated result.

For the rest of the proof see PJE's book. ■

Corollary 1.15 *For all $n \geq 1$ and $1 \leq r \leq n$,*

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

Proof Immediate from previous result and the definition of Binomial numbers.

■

The result of this corollary is usually represented as an unending triangle where each term, apart from those at the end of the rows, are the sum of the two terms in the line above.

Definition 1.16 *Pascal's Triangle.*

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
 & & & & \vdots & & & & \\
 & & & & \vdots & & & & \\
 & \binom{n-1}{0} & \cdots\cdots & \binom{n-1}{r-1} & & \binom{n-1}{r} & \cdots\cdots & \binom{n-1}{n-1} & \\
\binom{n}{0} & \cdots\cdots & \cdots\cdots & \cdots\cdots & \binom{n}{r} & \cdots\cdots & \cdots\cdots & \cdots\cdots & \binom{n}{n} \\
 & & & & \vdots & & & & \\
 & & & & \vdots & & & &
\end{array}
$$

The start of this is normally written as

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
 & & & & & & \vdots & & & & & & \\
 & & & & & & \vdots & & & & & &
\end{array}
$$

The last Corollary gives a result for $\binom{n}{r}$ in terms of $\binom{n-1}{r-1}$ and $\binom{n-1}{r}$. This is suitable for a proof by induction of the following

Theorem 1.17 *For all $n \geq 0$ and $0 \leq r \leq n$,*

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!},$$

with the convention that $0! = 1$.

Proof p.151 by induction ■

Theorem 1.18 ***Binomial Theorem****. Let $a, b \in \mathbb{R}$. For all $n \geq 1$ we have*

$$\begin{aligned}(a+b)^n &= \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i \\ &= a^n + na^{n-1}b + \binom{n}{2} a^{n-2}b^2 + ... \\ &\qquad ... + \binom{n}{n-2} a^2 b^{n-2} + nab^{n-1} + b^n,\end{aligned}$$

with the convention that $x^0 = 1$ for all $x \in \mathbb{R}$.

Proof p.153 by induction.

The induction step is based on assuming

$$(a+b)^k = \sum_{i=0}^{n} \binom{k}{i} a^{k-i} b^i$$

and using this within

$$(a+b)^{k+1} = (a+b)^k (a+b) = a\,(a+b)^k + b\,(a+b)^k.$$

■

Definition 1.19 *Because* $\binom{n}{r}$ *occur as coefficients in this expansion they are also known as the* ***binomial coefficients****.*

> **Advice for exam**, Theorem 1.18 is the Binomial Theorem, **not** Corollary 1.15 or Theorem 1.17.
>
> Also, the proof of Theorem 1.18 uses the ideas of relabelling in a sum, as in
>
> $$\sum_{i=0}^{n} a_{i+1} = \sum_{j=1}^{n+1} a_j = \sum_{i=1}^{n+1} a_i.$$
>
> This would not be a surprise in integrals
>
> $$\int_0^n f(t+1)\,dt = \int_1^{n+1} f(s)\,ds = \int_1^{n+1} f(t)\,dt,$$
>
> and so should not be a surprise in summations.

Example 1.20
- *From the sixth line in Pascal's triangle we see*

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$

- *The coefficient of* a^7b^3 *in* $(2a+b)^{10}$ *is*

$$2^7\binom{10}{3} = 2^7\frac{10!}{3!7!} = 128 \times \frac{10 \times 9 \times 8}{3 \times 2} = 15360.$$