A tropical approach to time stealing

Marianne Johnson, University of Manchester
(joint work with David Broomhead and Steve Furber)

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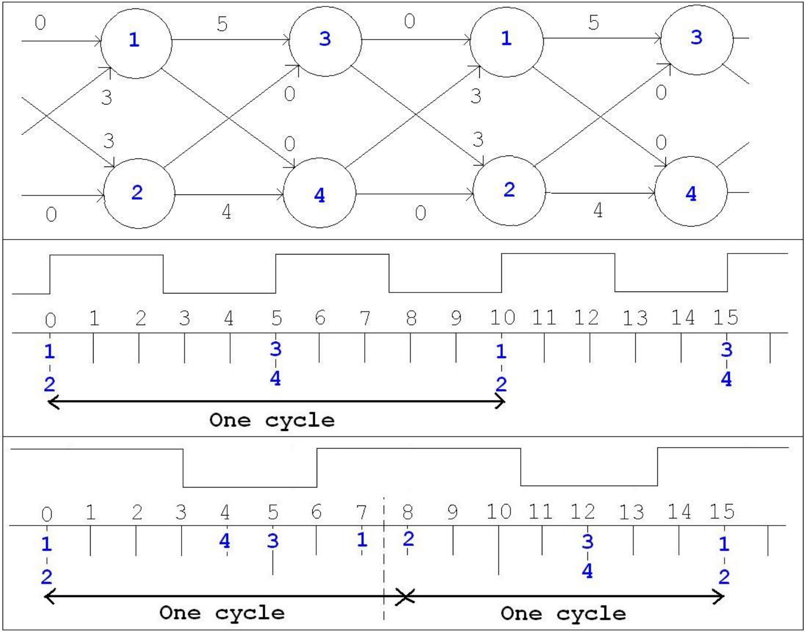
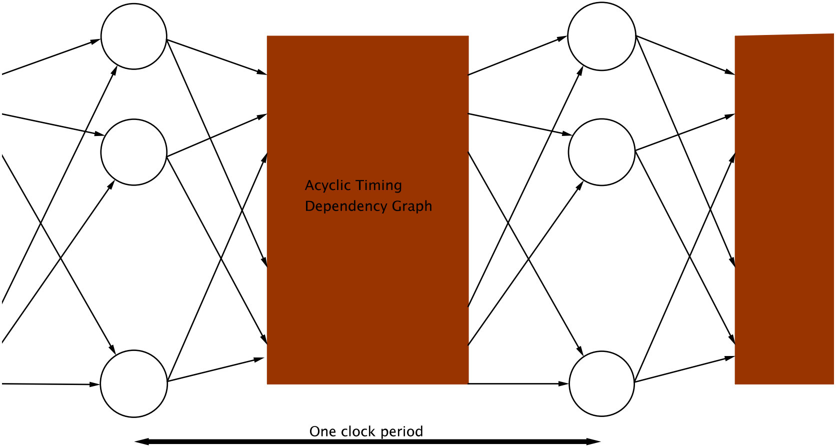
Synchronous and asynchronous circuits

- We are concerned with the timing of digital hardware.
- An obvious way to improve efficiency and utilization of resources is to let processes operate concurrently.
- In a **synchronous** circuit the input to each process is accepted on the rising (say) edge of the clock. Thus information is passed along at each tick of the clock.
- A more efficient use of time can occur in an **asynchronous** circuit. Here the input to each process will ideally be accepted as soon as all input signals have been received.
- Control is achieved using a multiphase clock; processes which are not permitted to operate concurrently are enabled at different phases of the clock.

Example



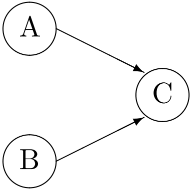
Generalisation



We consider timing dependency graphs with a periodic structure.

The algebra of timing

Imagine that a given process is waiting for inputs from two other processes.



The earliest time at which the input at C can be accepted is the **maximum** of the times at which the two input signals arrive. The input from A arrives at the time at which the input was accepted at A **plus** the delay time from A to C.

$$t_C = \max(t_A + d_A, t_B + d_B).$$

Thus, as we have already seen, in order to study the dynamics of this problem we need to consider the operations of maximisation and addition on the real numbers.

Tropical algebra

The **tropical semiring** has elements $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ and associative, commutative binary operations $\oplus$ and $\otimes$ defined by

$$a \oplus b = \max(a, b) \quad \text{and} \quad a \otimes b = a + b,$$

for all $a, b \in \mathbb{R}_{\max}$, where $\otimes$ distributes over $\oplus$.

The element $-\infty$ acts as a "zero" element, whilst the element 0 acts as a multiplicative identity. Thus for all $a \in \mathbb{R}_{\max}$:

$$\begin{aligned} a \oplus -\infty &= -\infty \oplus a = a, \\ a \otimes -\infty &= -\infty \otimes a = -\infty, \\ 0 \otimes a &= a \otimes 0 = a. \end{aligned}$$

For all $a \in \mathbb{R}_{\max}$ we also have $a \oplus a = a$. We say that $\mathbb{R}_{\max}$ is an **idempotent semiring**.

Tropical matrix algebra and directed graphs

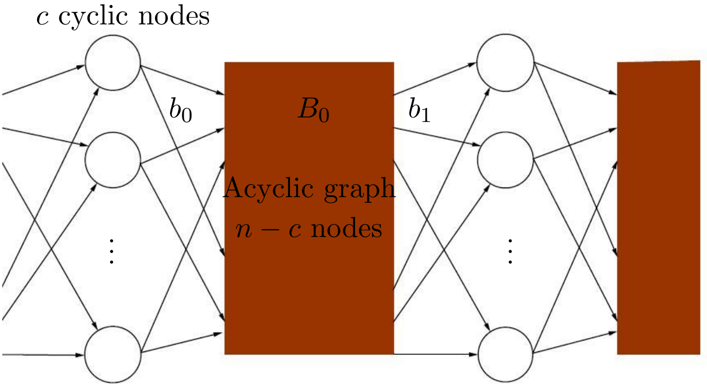
We define matrices over $\mathbb{R}_{\max}$ in the usual way.
The operations $\oplus$ and $\otimes$ can then be generalised as follows:

$$\begin{aligned}(A \oplus B)_{i,j} &= A_{i,j} \oplus B_{i,j}, \text{ for all } A, B \in \mathbb{R}_{\max}^{m \times n} \\ (A \otimes B)_{i,j} &= \bigoplus_{k=1}^{l} A_{i,k} \otimes B_{k,j}, \text{ for all } A \in \mathbb{R}_{\max}^{m \times l}, B \in \mathbb{R}_{\max}^{l \times n}.\end{aligned}$$

Given a finite weighted directed graph G on nodes $\{1, \ldots, n\}$ we associate to it an $n \times n$ matrix A as follows:

- If there is no edge from j to i then $A_{i,j} = -\infty$;
- If there is an edge from j to i labelled by w then $A_{i,j} = w$.

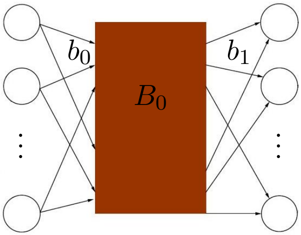
Matrix of delay times



- Identify the 'cyclic nodes'.
- The matrix of delay times is

$$A = \left(\begin{array}{c|c} -\infty & b_1 \\ \hline b_0 & B_0 \end{array} \right)$$

Dynamics



$$A = \left(\begin{array}{c|c} -\infty & b_1 \\ \hline b_0 & B_0 \end{array}\right)$$

where b_0 is $(n-c) \times c$,
B_0 is $(n-c) \times (n-c)$,
b_1 is $c \times (n-c)$.

- For $i = 1, \ldots, n$ let $x_i(k)$ denote the time at which process i accepts its input for the kth time.
- To get going we need an initial condition.
  Suppose we know $x_1(1), \ldots, x_n(1)$.
- Let $A_0 = \left(\begin{array}{c|c} -\infty & -\infty \\ \hline b_0 & B_0 \end{array}\right)$ and $A_1 = \left(\begin{array}{c|c} -\infty & b_1 \\ \hline -\infty & -\infty \end{array}\right)$.
- Then $A = A_0 \oplus A_1$ and

$$x(k) = (A_0 \otimes x(k)) \oplus (A_1 \otimes x(k-1)).$$

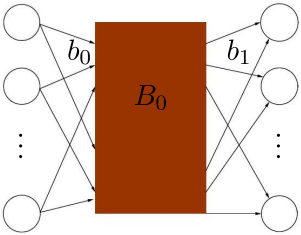
The Kleene star

Given an $n \times n$ matrix of delay times A corresponding to an **acyclic graph** G, the **Kleene star** $A^\star$ is defined as

$$A^\star = \bigoplus_{k \geqslant 0} A^{\otimes k}.$$

- $A^{\otimes k}_{i,j}$ gives the maximum delay of paths of length k in G from j to i.
- Since the graph is acyclic, $A^\star$ is given by a sum of a finite number of terms.
- $A^\star_{i,j}$ gives the maximum delay of paths in G from j to i.
- By substitution it is easy to check that $x = A^\star \otimes b$ is a solution of $x = (A \otimes x) \oplus b$.

Dynamics



$$A = \left(\begin{array}{c|c} -\infty & b_1 \\ \hline b_0 & B_0 \end{array}\right),\ A = A_0 \oplus A_1 \text{ where}$$

$$A_0 = \left(\begin{array}{c|c} -\infty & -\infty \\ \hline b_0 & B_0 \end{array}\right),\ A_1 = \left(\begin{array}{c|c} -\infty & b_1 \\ \hline -\infty & -\infty \end{array}\right).$$

- Recall that $x_i(k)$ is the time at which process i accepts its input for the kth time.
- Initial condition: $x_1(1), \ldots, x_c(1), x_{c+1}, \ldots, x_n(1)$.
- Then

$$x(k) = (A_0 \otimes x(k)) \oplus (A_1 \otimes x(k-1)).$$

- Thus, using the Kleene star, we find

$$x(k) = A_0^\star \otimes A_1 \otimes x(k-1).$$

The form of our Kleene star

The dynamics of our system are governed by the system of equations

$$x(k) = A_0^\star \otimes A_1 \otimes x(k-1).$$

Recall that the matrices A_0 and A_1 have a nice block matrix form, with lots of $-\infty$ entries. Using this nice block matrix structure it is then easy to check that:

$$\begin{aligned} A_0^\star &= \left(\begin{array}{c|c} \mathrm{id} & -\infty \\ \hline B_0^\star \otimes b_0 & B_0^\star \end{array}\right), \\ A_0^\star \otimes A_1 &= \left(\begin{array}{c|c} -\infty & b_1 \\ \hline -\infty & B_0^\star \otimes b_0 \otimes b_1 \end{array}\right) \end{aligned}$$

So we only need to know $x_{c+1}(k-1), \ldots, x_n(k-1)$.

Eigenvalues and minimum clock period

Given an $n \times n$ matrix A we look for a $\lambda \in \mathbb{R}_{\max}$ and a (non-trivial) vector $x \in \mathbb{R}_{\max}^n$ such that

$$A \otimes x = \lambda \otimes x.$$

Theorem If G_A is strongly connected then A possesses a unique eigenvalue. Moreover, the eigenvalue is the real number (i.e. not $-\infty$) equal to the maximal average delay of circuits in G_A.

It can be shown that the minimum clock period is given by the eigenvalue of $B_0^\star \otimes b_0 \otimes b_1$, and that this coincides with the heuristic given by the ARM designers.