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Standard Tableaux and Klyachko's Theorem

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Partitions, Standard Tableaux and the Major Index

- We say $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a **partition** of n ($\lambda \vdash n$) if

$$\lambda_1 \geq \cdots \geq \lambda_k > 0 \quad \text{and} \quad \lambda_1 + \cdots + \lambda_k = n$$

We call the λ_i the **parts** of λ. For partitions with many equal parts it is convenient to write

$$\lambda = (\lambda_1^{n_1}, \ldots, \lambda_l^{n_l}) \quad \lambda_1 > \cdots > \lambda_l > 0$$

to denote that there are exactly n_i parts equal to λ_i.

- The **Young diagram** of shape λ is a collection of n boxes arranged in left justified rows with λ_i boxes in the ith row.
- A **tableau** of shape λ is a filling of the Young diagram of λ with the numbers from $\{1, \ldots, k : k \leq n\}$ such that
  - entries weakly increase along rows
  - entries strictly increase down columns.

  We say that a tableau is **standard** if each number in $\{1, \ldots, n\}$ occurs exactly once.
- An entry i in a standard tableau is called a **descent** if $i+1$ occurs in any row below i.
- We define the sum of all descents in a standard tableau T to be the **major index** of T, $\mathrm{maj}(T)$.

A Young diagram of shape $\lambda = (5, 3, 2, 1)$

1	1	1	2	4
2	2	3		
3	4			
4				

A tableau

1	**2**	**4**	8	**9**
3	**5**	11		
6	10			
7				

A standard tableau

$$\mathrm{maj}(T) = 2 + 4 + 5 + 6 + 9 = 26$$

Theorem [1]

Let $n \geq 3$, $\lambda \vdash n$.
There exists a standard tableau of shape λ with major index coprime to n if and only if $\lambda \neq (1^n), (n), (2^2)$ or (2^3).

Proof

It is easy to prove the Theorem in one direction. Indeed, shown below are all the standard tableaux of shape $(1^n), (n), (2^2)$ or (2^3). Clearly, none of these have major index which is coprime to n.

1
2
n

$\mathrm{maj}(T) = \frac{n(n-1)}{2}$

1	2		n

$\mathrm{maj}(T) = 0$

1	**2**
3	4

$\mathrm{maj}(T) = 2$

1	**3**
2	4

$\mathrm{maj}(T) = 4$

1	**2**
3	**4**
5	6

$\mathrm{maj}(T) = 6$

1	**2**
3	**5**
4	6

$\mathrm{maj}(T) = 10$

1	**3**
2	**4**
5	6

$\mathrm{maj}(T) = 8$

1	**3**
2	**5**
4	6

$\mathrm{maj}(T) = 9$

1	**4**
2	**5**
3	6

$\mathrm{maj}(T) = 12$

The proof in the opposite direction is less obvious.

Key Idea: Small descent sets

We look at standard tableaux with "small" descent sets. Let T be a standard tableau of shape $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ and let d_T denote the number of descents in T. Then $k - 1 \leq d_T \leq n - \lambda_1$. To see this it is useful to look at the example below.

1	2	3	4	**5**
6	7	**8**		
9	**10**			
11				

$k - 1$ descents

1	**5**	**8**	10	11
2	**6**	9		
3	7			
4				

$n - \lambda_1$ descents

It turns out that, if λ is a partition of $n \geq 3$ with k parts, $\lambda \neq (1^n), (n), (2^2)$ or (2^3), then there exists a standard tableau of shape λ with at most k descents which has major index coprime to n.

Lie Representations

- Let V be a finite dimensional vector space over a field of characteristic zero and denote by $T = T(V)$ the tensor algebra of V

$$T = \bigoplus_{n \geq 0} T_n \quad \text{where} \quad T_n = V^{\otimes n}.$$

- Each tensor power T_n is a semisimple module for the general linear group $GL(V)$. Moreover, the isomorphism types of the irreducible submodules of T_n are parameterised by partitions of n with at most $\dim(V)$ parts. We have that

$$T_n = \bigoplus_{\lambda \vdash n} t_\lambda V_\lambda, \quad V_\lambda \ \text{irreducible}, \quad 0 \leq t_\lambda.$$

- Consider T as a Lie algebra via the multiplication $[x, y] = x \otimes y - y \otimes x$. By a theorem of Witt the Lie subalgebra $L = L(V)$ generated by V in T is the free Lie algebra on V.

$$L = \bigoplus_{n \geq 1} L_n$$

where each Lie power $L_n = T_n \cap L$ is a $GL(V)$-submodule of T_n.

- Hence, the isomorphism types of the irreducible $GL(V)$-submodules of L_n form a subset of those occurring in T_n.

$$L_n = \bigoplus_{\lambda \vdash n} l_\lambda V_\lambda, \quad V_\lambda \ \text{irreducible}, \quad 0 \leq l_\lambda \leq t_\lambda.$$

- It is natural to wonder when the multiplicities t_λ and l_λ are non-zero.

When are the multiplicities non-zero?

It turns out that the multiplicity $t_\lambda \geq 1$ for all partitions λ. That is, every irreducible $GL(V)$ module occurs in the module decomposition of T_n. In 1949 Wever [4] gave a formula for computing the multiplicities l_λ which involved calculating certain character values. However, it is not immediately obvious from this formula for which λ we have $l_\lambda \geq 1$.

In 1974 Klyachko proved that almost all of the irreducible modules ocur in the decomposition of L_n.

Klyachko's Theorem [2]

Let $n \geq 3$, $\lambda \vdash n$.
$l_\lambda \geq 1$ if and only if λ has no more than $\dim(V)$ parts and $\lambda \neq (1^n), (n), (2^2), (2^3)$.

What does this have to do with standard tableaux?

In 1987 Kraśkiewicz and Weyman gave a combinatorial interpretation of the multiplicities. They proved that the multiplicity l_λ can be calculated by counting the number of standard tableaux with a particular major index.

Kraśkiewicz-Weyman Theorem [3]

Let $a, n \in \mathbb{N}$ be fixed coprime numbers, $\lambda \vdash n$ with at most $\dim(V)$ parts.

The multiplicity l_λ is equal to the number of standard tableaux of shape λ with major index congruent to a modulo n.

Theorem ⇔ Klyachko's Theorem

It has been a longstanding challenge to deduce Klyachkos Theorem directly from the Kraśkiewicz-Weyman Theorem. It is easy to see that our Theorem (left) together with the Kraśkiewicz-Weyman Theorem imply Klyachko's Theorem.

References

[1] Marianne Johnson. Standard tableaux and Klyachko's Theorem on Lie representations. *Journal of Combinatorial Theory, Series A*(to appear).

[2] A.A. Klyachko. Lie elements in the tensor algebra. *Sibirsk Mat. Ž*, 15:12961304, 1974. (Russian). English translation: *Siberian J. Math.* **15** (1974), 914-921.

[3] Witold Kraśkiewicz and Jerzy Weyman. Algebra of coinvariants and the action of a Coxeter element. *Bayreuth. Math. Schr.*, 63:265284, 2001. (Preprint, 1987).

[4] F. Wever. Über Invarianten von Lieschen Ringen. *Math. Annalen*, 120:563580, 1949.