On torsion in free central extensions of groups

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Free centre-by-metabelian groups

- Let $F = F(X)$ be the free group on a set X.
- The quotient
$$F/[F'', F]$$
is the free centre-by-metabelian group on X.
- In view of the short exact sequence
$$1 \to F''/[F'', F] \to F/[F'', F] \to F/F'' \to 1$$
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Kanta Gupta's discovery

Theorem (C.K. Gupta, 1973)

The free centre-by-metabelian group $F/[F'', F]$ of rank n is torsion free for $n = 2, 3$, and for $n \geq 4$ it contains an elementary abelian 2-group of rank $\binom{n}{4}$ in its centre.

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Kuzmin's observation

The Crucial Observation (Kuzmin, 1977)

The torsion subgroup of $F''/[F'', F]$ is isomorphic to the integral homology group $H_4(F/F')$ reduced modulo 2,

$$t(F''/[F'', F]) = H_4(F/F') \otimes \mathbb{Z}_2.$$

- $F/[F'', F] = F/[\gamma_2(F'), F]$ where $\gamma_2(F')$ is the second term of the lower central series of F'.
- Let R be any normal subgroup of F, $\gamma_c R$ an arbitrary term of the lower central series of R
- We get a free central extension $F/[\gamma_c R, F]$, of which the free centre-by-metabelian group is a special case...

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The problem

- Let G be a group given by a free presentation $G = F/R$ and consider the quotient $F/[\gamma_c(R), F]$, $(c \geq 2)$.
- In view of the short exact sequence

$$1 \to \gamma_c(R)/[\gamma_c(R), F] \to F/[\gamma_c(R), F] \to F/\gamma_c(R) \to 1$$

$F/[\gamma_c(R), F]$ is a free central extension of the group $F/\gamma_c(R)$, which is in turn an extension of $G = F/R$ with free nilpotent kernel:

$$1 \to R/\gamma_c(R) \to F/\gamma_c(R) \to F/R \to 1.$$

- While $F/\gamma_c(R)$ is always torsion free (Shmelkin, 1965), elements of finite order may occur in the central quotient $\gamma_c(R)/[\gamma_c(R), F]$.

Q: What is the torsion subgroup of $\gamma_c(R)/[\gamma_c(R), F]$?

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Relation modules, Lie rings and Trivialization

- The abelianization $R_{ab} = R/R'$ is a module for the group $G = F/R$ (with action given by conjugation) called the **relation module**.
- For an arbitrary $\mathbb{Z}$-free G-module V, let $L(V)$ denote the **free Lie ring** on V. This is a graded Lie ring,

$$L(V) = \bigoplus_{n \geq 1} L_n(V)$$

where $L_n(V)$ is the nth homogeneous component of $L(V)$.

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- There is a classical isomorphism of an G-modules, $\gamma_c(R)/\gamma_{c+1}(R) \cong L_c(R_{ab})$,
- Trivialising the G-action gives the following lemma:

Lemma (Baumslag, Strebel and Thomson, 1980)

Let $G = F/R$. Then there is an isomorphism

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In order to find the torsion subgroup of $\gamma_c(R)/[\gamma_c(R), F]$, we can concentrate on

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The Exponent Theorem

Theorem (Kuzmin, 1982; Stöhr, 1987)

Let R be an arbitrary normal subgroup of F. Then the torsion subgroup $t_c = t(\gamma_c(R)/[\gamma_c(R), F])$ is of exponent dividing c if $c \geq 3$, and of exponent dividing 4 if $c = 2$:

$$\begin{aligned} c\ t_c &= 0 \quad ,\textit{for} \quad c \geq 3, \\ \textit{and} \quad 4\ t_2 &= 0. \end{aligned}$$

Exact results

Theorem (Stöhr, 1987)

Let p be a prime, and let R be a normal subgroup of F such that $G = F/R$ has no elements of order p. Then

$$t_p \cong H_4(G, \mathbb{Z}_p).$$

Theorem (Stöhr, 1993)

If $G = F/R$ has no elements of order 2, then

$$t_4 \cong H_6(G, \mathbb{Z}_2).$$

The Bryant-Schocker Decomposition Theorem

New results in the theory of modular Lie powers have made it possible to make further progress. Most importantly...

Theorem (Bryant and Schocker, 2006)

Let K be a field of characteristic p, G a group, V a KG-module and k a positive integer not divisible by p. Then

$$L_{p^m k}(V) \cong L_{p^m}(B_k) \oplus L_{p^{m-1}}(B_{pk}) \oplus \cdots \oplus L_1(B_{p^m k}),$$

where the modules $B_{p^i k}$ satisfy

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New Developments

- Want to identify the torsion subgroup of $L_c(R_{ab}) \otimes_G \mathbb{Z}$.
- Consider the short exact sequence of G-modules

$$0 \longrightarrow L_c(R_{ab}) \xrightarrow{p} L_c(R_{ab}) \longrightarrow L_c(R_{ab}) \otimes \mathbb{Z}_p \longrightarrow 0.$$

- Part of the associated long exact homology sequence is

$$\rightarrow H_1(G, L_c(R_{ab}) \otimes \mathbb{Z}_p) \rightarrow L_c(R_{ab}) \otimes_G \mathbb{Z} \xrightarrow{p} L_c(R_{ab}) \otimes_G \mathbb{Z} \rightarrow$$

- So the homology group on the left is the key to the elements of order p in $L_c(R_{ab}) \otimes_G \mathbb{Z}$.

We will use the Bryant-Schocker Decompostion Theorem to calculate $H_1(G, L_c(R_{ab}) \otimes \mathbb{Z}_p)$ for $c = 6$, $p = 2, 3$

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- Let $p = 2,\ k = 3,\ K = \mathbb{Z}_2,\ V = R_{ab} \otimes \mathbb{Z}_2$.
- Then the Bryant-Schocker Decomposition Theorem gives that **modulo** 2

$$L_6(R_{ab}) \cong L_2(L_3(R_{ab})) \oplus B_6^{(2)}$$

and

$$2B_6^{(2)} \oplus T_2(L_3(R_{ab})) \cong L_3(T_2(R_{ab})).$$

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- $L_3(R_{ab})$ is a projective $\mathbb{Z}_2 G$-module (provided that G has no elements of order 3),
- $L_2(L_3(R_{ab}))$ is also a projective $\mathbb{Z}_2 G$-module (provided that G has no elements of order 2)
- Can show that $L_3(T_2(R_{ab}))$ has trivial homology in all dimensions ≥ 1
- hence $B_6^{(2)}$ has trivial homology in all dimensions ≥ 1

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and hence there are no elements of order 2 in $L_6(R_{ab}) \otimes_G \mathbb{Z}$ (provided G has no 2-torsion and no 3-torsion).

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=& H_1(G, L_2(L_3(R_{ab})) \otimes \mathbb{Z}_2) \oplus H_1(G, B_6^{(2)}) \\
=& 0,
\end{aligned}
$$

and hence there are no elements of order 2 in $L_6(R_{ab}) \otimes_G \mathbb{Z}$ (provided G has no 2-torsion and no 3-torsion).

- Let $p = 3, \; k = 2, \; K = \mathbb{Z}_3, \; V = R_{ab} \otimes \mathbb{Z}_3$.
- Then the Bryant-Schocker Decomposition Theorem gives that **modulo** 3

$$L_6(R_{ab}) \cong L_3(L_2(R_{ab})) \oplus B_6^{(3)}$$

and

$$3B_6^{(3)} \oplus T_3(L_2(R_{ab})) \cong L_2(T_3(R_{ab})).$$

By arguing similarly to the case $p = 2$ we obtain...

$$\begin{aligned} & H_1(G, L_6(R_{ab}) \otimes \mathbb{Z}_3) \\ =& H_1(G, L_3(L_2(R_{ab})) \otimes \mathbb{Z}_3) \oplus H_1(G, B_6^{(3)}) \\ =& 0, \end{aligned}$$

and hence there are no elements of order 3 in $L_6(R_{ab}) \otimes_G \mathbb{Z}$ (provided G has no 2-torsion and no 3-torsion).

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$c = 6$

So if G has no elements of order 2 and order 3 we have shown that the abelian group $L_6(R_{ab}) \otimes_G \mathbb{Z}$ has no elements of order 2 and no elements of order 3. However, by the Exponent Theorem, the torsion subgroup of this abelian group is of exponent dividing 6. **Therefore it has no elements of finite order at all.**

Theorem (R. Stöhr and M.J.)

Let $G = F/R$ be a group without elements of order 2 and order 3, then the group $F/[\gamma_6(R), F]$ is torsion-free.

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Exact results known to date

If $G = F/R$ *has no non-trivial elements of order dividing* c *then*

$$t_c \cong \begin{cases} H_4(G, \mathbb{Z}_p), & \text{if } c = p,\ p \text{ a prime;} \\ H_6(G, \mathbb{Z}_2), & \text{if } c = 4; \\ 0, & \text{if } c = 6. \end{cases}$$