On torsion in free central extensions of groups

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Free centre-by-metabelian groups

- Let $F = F(X)$ be the free group on a set X.
- The quotient

$$F/[F'', F]$$

is the free centre-by-metabelian group on X.
- In view of the short exact sequence

$$1 \to F''/[F'', F] \to F/[F'', F] \to F/F'' \to 1$$

this is a free central extension of the free metabelian group F/F''.

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Kanta Gupta's discovery

Theorem (C.K. Gupta, 1973)

The free centre-by-metabelian group $F/[F'', F]$ of rank n is torsion free for $n = 2, 3$, and for $n \geq 4$ it contains an elementary abelian 2-group of rank $\binom{n}{4}$ in its centre.

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Kuz'min's observation

The Crucial Observation (Kuz'min, 1977)

The torsion subgroup of $F''/[F'', F]$ is isomorphic to the integral homology group $H_4(F/F')$ reduced modulo 2,

$$t(F''/[F'', F]) = H_4(F/F') \otimes \mathbb{Z}_2.$$

- $F/[F'', F] = F/[\gamma_2(F'), F]$ where $\gamma_2(F')$ is the second term of the lower central series of F'.
- Let R be any normal subgroup of F, $\gamma_c R$ an arbitrary term of the lower central series of R.
- We get a free central extension $F/[\gamma_c R, F]$, of which the free centre-by-metabelian group is a special case...

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The problem

- Let G be a group given by a free presentation $G = F/R$ and consider the quotient $F/[\gamma_c(R), F]$, $(c \geq 2)$.
- In view of the short exact sequence

$$1 \to \gamma_c(R)/[\gamma_c(R), F] \to F/[\gamma_c(R), F] \to F/\gamma_c(R) \to 1$$

$F/[\gamma_c(R), F]$ is a free central extension of the group $F/\gamma_c(R)$, which is in turn an extension of $G = F/R$ with free nilpotent kernel:

$$1 \to R/\gamma_c(R) \to F/\gamma_c(R) \to F/R \to 1.$$

- While $F/\gamma_c(R)$ is always torsion free (Shmelkin, 1965), elements of finite order may occur in the central quotient $\gamma_c(R)/[\gamma_c(R), F]$.

Q: What is the torsion subgroup of $\gamma_c(R)/[\gamma_c(R), F]$?

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Relation modules, Lie rings and Trivialization

- The abelianization $R_{ab} = R/R'$ is a module for the group $G = F/R$ (with action given by conjugation) called the **relation module**.
- For an arbitrary $\mathbb{Z}$-free G-module V, let $L(V)$ denote the **free Lie ring** on V. This is a graded Lie ring,

$$L(V) = \bigoplus_{n \geq 1} L_n(V)$$

where $L_n(V)$ is the nth homogeneous component of $L(V)$.

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Relation modules, Lie rings and Trivialization (cont.)

- There is a classical isomorphism of G-modules,

$$\gamma_c(R)/\gamma_{c+1}(R) \cong L_c(R_{ab}).$$

- Trivializing the G-action on both sides gives:

$$(\gamma_c(R)/\gamma_{c+1}(R)) \otimes_G \mathbb{Z} \cong L_c(R_{ab}) \otimes_G \mathbb{Z}.$$

Lemma (Baumslag, Strebel and Thomson, 1980)

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The Exponent Theorem

Theorem (Kuz'min, 1982; Stöhr, 1987)

Let R be an arbitrary normal subgroup of F. Then the torsion subgroup $t_c = t(\gamma_c(R)/[\gamma_c(R), F])$ is of exponent dividing c if $c \geq 3$, and of exponent dividing 4 if $c = 2$:

$$\begin{aligned} c\ t_c &= 0, \quad \text{for} \quad c \geq 3, \\ \text{and} \quad 4\ t_2 &= 0. \end{aligned}$$

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Exact results

Theorem (Stöhr, 1987)

Let p be a prime, and let R be a normal subgroup of F such that $G = F/R$ has no elements of order p. Then

$$t_p \cong H_4(G, \mathbb{Z}_p).$$

Theorem (Stöhr, 1993)

If $G = F/R$ has no elements of order 2, then

$$t_4 \cong H_6(G, \mathbb{Z}_2).$$

The Bryant-Schocker Decomposition Theorem

New results in the theory of modular Lie powers have made it possible to make further progress. Most importantly...

Theorem (Bryant and Schocker, 2006)

Let K be a field of characteristic p, G a group, V a KG-module and k a positive integer not divisible by p. Then

$$L_{p^m k}(V) \cong L_{p^m}(B_k) \oplus L_{p^{m-1}}(B_{pk}) \oplus \cdots \oplus L_1(B_{p^m k}),$$

where the modules $B_{p^i k}$ satisfy

$$p^m B_{p^m k} \oplus p^{m-1} T_p(B_{p^{m-1} k}) \oplus \cdots \oplus T_{p^m}(B_k) \cong L_k(T_{p^m}(V)).$$

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New Developments

- Want to identify the torsion subgroup of $L_c(R_{ab}) \otimes_G \mathbb{Z}$.
- Consider the short exact sequence of G-modules

$$0 \longrightarrow L_c(R_{ab}) \xrightarrow{p} L_c(R_{ab}) \longrightarrow L_c(R_{ab}) \otimes \mathbb{Z}_p \longrightarrow 0.$$

- Part of the associated long exact homology sequence is

$$\to H_1(G, L_c(R_{ab}) \otimes \mathbb{Z}_p) \to L_c(R_{ab}) \otimes_G \mathbb{Z} \xrightarrow{p} L_c(R_{ab}) \otimes_G \mathbb{Z} \to$$

- So the homology group on the left is the key to the elements of order p in $L_c(R_{ab}) \otimes_G \mathbb{Z}$.

We will use the Bryant-Schocker Decomposition Theorem to calculate $H_1(G, L_c(R_{ab}) \otimes \mathbb{Z}_p)$ *for* $c = 6$, $p = 2, 3$

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- A special case of the Bryant-Schocker Decomposition Theorem (take $K = \mathbb{Z}_2$, $V = R_{ab} \otimes \mathbb{Z}_2$, with $p = 2$, $k = 3$ and $m = 1$) gives that

$$L_6(R_{ab}) \otimes \mathbb{Z}_2 \cong [L_2(L_3(R_{ab})) \otimes \mathbb{Z}_2] \oplus B_6^{(2)}$$

and

$$2B_6^{(2)} \oplus [T_2(L_3(R_{ab})) \otimes \mathbb{Z}_2] \cong L_3(T_2(R_{ab})) \otimes \mathbb{Z}_2.$$

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- Thus $L_2(L_3(R_{ab})) \otimes \mathbb{Z}_2$ is also a projective $\mathbb{Z}_2 G$-module (provided that, additionally, G has no elements of order 2).
- One can show that $L_3(T_2(R_{ab}))$ has trivial homology in all dimensions ≥ 1.
- Hence, $B_6^{(2)}$ has trivial homology in all dimensions ≥ 1.

Consequently,

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and hence there are no elements of order 2 *in* $L_6(R_{ab}) \otimes_G \mathbb{Z}$ *(provided* G *has no* 2*-torsion and no* 3*-torsion).*

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$$\begin{aligned} & H_1(G, L_6(R_{ab}) \otimes \mathbb{Z}_2) \\ = & H_1(G, L_2(L_3(R_{ab})) \otimes \mathbb{Z}_2) \oplus H_1(G, B_6^{(2)}) \\ = & 0, \end{aligned}$$

and hence there are no elements of order 2 in $L_6(R_{ab}) \otimes_G \mathbb{Z}$ (provided G has no 2-torsion and no 3-torsion).

$c = 6,\ p = 2$

- $L_3(R_{ab}) \otimes \mathbb{Z}_2$ is a projective $\mathbb{Z}_2 G$-module (provided that G has no elements of order 3).
- Thus $L_2(L_3(R_{ab})) \otimes \mathbb{Z}_2$ is also a projective $\mathbb{Z}_2 G$-module (provided that, additionally, G has no elements of order 2).
- One can show that $L_3(T_2(R_{ab}))$ has trivial homology in all dimensions ≥ 1.
- Hence, $B_6^{(2)}$ has trivial homology in all dimensions ≥ 1.

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$c = 6,\ p = 3$

- Another special case of the Bryant-Schocker Decomposition Theorem (this time take $K = \mathbb{Z}_3$, $V = R_{ab} \otimes \mathbb{Z}_3$, with $p = 3$, $k = 2$ and $m = 1$) gives that

$$L_6(R_{ab}) \otimes \mathbb{Z}_3 \cong [L_3(L_2(R_{ab})) \otimes \mathbb{Z}_3] \oplus B_6^{(3)}$$

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So if G has no elements of order 2 and order 3 we have shown that the abelian group $L_6(R_{ab}) \otimes_G \mathbb{Z}$ has no elements of order 2 and no elements of order 3. However, by the Exponent Theorem, the torsion subgroup of this abelian group is of exponent dividing 6. **Therefore it has no elements of finite order at all.**

Theorem (R. Stöhr and M.J.)

Let $G = F/R$ be a group without elements of order 2 and order 3, then the group $F/[\gamma_6(R), F]$ is torsion-free.

See MIMS eprint 2008.51.

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Exact results known to date

If $G = F/R$ has no non-trivial elements of order dividing c then

$$t_c \cong \begin{cases} H_4(G, \mathbb{Z}_p), & \text{if } c = p,\ p \text{ a prime;} \\ H_6(G, \mathbb{Z}_2), & \text{if } c = 4; \\ 0, & \text{if } c = 6. \end{cases}$$

Example: Free centre-by-(nilpotent-by-abelian) groups

- The free centre-by-(nilpotent-by-abelian) group $F/[\gamma_c(F'), F]$ contains non-trivial elements of finite order if $c = 2, 3, 4, 5, 7, 11, \ldots$
- that is if c is a prime or $c = 4$, (except for small ranks)
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Recall the Exponent Theorem:

Theorem (Kuz'min, 1982; Stöhr, 1987)

Let R be an arbitrary normal subgroup of F. Then the torsion subgroup $t_c = t(\gamma_c(R)/[\gamma_c(R), F])$ is of exponent dividing c if $c \geq 3$, and of exponent dividing 4 if $c = 2$:

$$\begin{aligned} c\ t_c &= 0 \quad , \text{for} \quad c \geq 3, \\ \text{and} \quad 4\ t_2 &= 0. \end{aligned}$$

If $c = pq^m$ where q is prime, $p \neq q$, and $G = F/R$ has no non-trivial elements of order dividing c then

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Some partial results

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