Adams operations and Lie resolvents

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The Green ring

G a group, K a field. Consider finite-dimensional right KG-modules. Take char $K = p$ (usually prime).

The **Green ring** (or representation ring) R_{KG} has $\mathbb{Z}$-basis the isomorphism classes of (f. d.) indecomposable KG-modules with multiplication coming from tensor product.

KG-modules	$U \oplus V$	$U \otimes_K V$	$V^{\otimes n}$
R_{KG}	$U + V$	UV	V^n

Example. $G = C_{p^m} = \langle a \rangle$. For $r = 1, \ldots, p^m$, $KG(a-1)^r$ is a submodule of KG. Write $V_r = KG/KG(a-1)^r$. Then V_r is indecomposable of dimension r. R_{KG} has $\mathbb{Z}$-basis $\{V_1, \ldots, V_{p^m}\}$. Green (1962) showed how to multiply in R_{KG}.

We can extend R_{KG} to $\mathbb{Q} \otimes R_{KG}$ by allowing coefficients in $\mathbb{Q}$ rather than $\mathbb{Z}$.

Symmetric, exterior and Lie powers

Let V be a K-space with basis $\{x_1, \ldots, x_r\}$. Write
$S(V) = K[x_1, \ldots, x_r]$ (free assoc. comm. K-algebra),
$\Lambda(V) =$ free assoc. K-algebra on $x_1, \ldots, x_r$ subject to
$x_i \wedge x_i = 0$ and $x_i \wedge x_j = -x_j \wedge x_i$,
$L(V) =$ free Lie algebra over K on $x_1, \ldots, x_r$.
Take decompositions into homogeneous components:
$S(V) = S^0(V) \oplus S^1(V) \oplus \cdots \oplus S^n(V) \oplus \cdots$,
$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \cdots \oplus \Lambda^n(V) \oplus \cdots$,
$L(V) = L^1(V) \oplus \cdots \oplus L^n(V) \oplus \cdots$.
These components are the 'symmetric powers', 'exterior powers' and 'Lie powers' of V. If V is a KG-module then $S^n(V)$, $\Lambda^n(V)$ and $L^n(V)$ become KG-modules by linear substitutions.
$S^0(V) \cong \Lambda^0(V) \cong K$, written as 1 in R_{KG}.
$S^1(V) \cong \Lambda^1(V) \cong L^1(V) \cong V$.

General problem. Determine $S^n(V)$, $\Lambda^n(V)$ and $L^n(V)$ up to isom. as sums of indecomposables, i.e. as elements of R_{KG}.

Adams operations

Consider the power series ring $(\mathbb{Q} \otimes R_{KG})[[t]]$. Define $\psi_S^n(V)$ and $\psi_\Lambda^n(V)$ in $\mathbb{Q} \otimes R_{KG}$ by

$$\psi_S^1(V)t + \tfrac{1}{2}\psi_S^2(V)t^2 + \tfrac{1}{3}\psi_S^3(V)t^3 + \cdots = \log(1 + S^1(V)t + S^2(V)t^2 + \cdots),$$

$$\psi_\Lambda^1(V)t - \tfrac{1}{2}\psi_\Lambda^2(V)t^2 + \tfrac{1}{3}\psi_\Lambda^3(V)t^3 - \cdots = \log(1 + \Lambda^1(V)t + \Lambda^2(V)t^2 + \cdots).$$

It turns out that $\psi_S^n(V), \psi_\Lambda^n(V) \in R_{KG}$ and

$$\psi_S^n(U+V) = \psi_S^n(U) + \psi_S^n(V), \quad \psi_\Lambda^n(U+V) = \psi_\Lambda^n(U) + \psi_\Lambda^n(V).$$

Thus we get $\mathbb{Z}$-linear functions, called **Adams operations**,

$$\psi_S^n, \psi_\Lambda^n : R_{KG} \to R_{KG}.$$

Properties of Adams operations

Clearly $\psi_S^1(V), \ldots, \psi_S^n(V)$ are polynomials in $S^1(V), \ldots, S^n(V)$ and vice versa. Thus knowledge of symmetric powers in R_{KG} is equivalent to knowledge of the Adams operations (assuming we know how to multiply in R_{KG}). Similarly for exterior powers.

Problem. For given G and K determine ψ_S^n and ψ_Λ^n.

Note that ψ_S^n and ψ_Λ^n are linear, whereas S^n and Λ^n are not. Results for ψ_S^n and ψ_Λ^n are often 'simpler' than for S^n and Λ^n.

The main properties of the Adams operations on R_{KG} were given by Benson (1984) and RMB (2003) following ideas of Adams, Frobenius and others.

Good behaviour when n is not divisible by p: for $p \nmid n$, $\psi_S^n = \psi_\Lambda^n$, and ψ_S^n is a ring endomorphism of R_{KG}.

Factorisation (RMB, 2003): if $n = kp^d$ where $p \nmid k$ then

$$\psi_S^n = \psi_S^k \circ \psi_S^{p^d}, \;\; \psi_\Lambda^n = \psi_\Lambda^k \circ \psi_\Lambda^{p^d}.$$

Lie resolvents

Theorem (RMB, 2003). There are $\mathbb{Z}$-linear functions $\Phi^n : R_{KG} \to R_{KG}$ such that

$$L^n(V) = \frac{1}{n} \sum_{d|n} \Phi^d(V^{n/d})$$

and (by Möbius inversion)

$$\Phi^n(V) = \sum_{d|n} \mu(n/d)\, d\, L^d(V^{n/d})$$

for all f.d. KG-modules V and all positive integers n.

Here μ denotes the Möbius function.
Thus knowledge of $\{L^d : d \mid n\}$ is equivalent to knowledge of $\{\Phi^d : d \mid n\}$.

Problem. For given G and K determine Φ^n.

Properties of Lie resolvents

Results for Φ^n are often simpler in form than the corresponding results for L^n.

Good behaviour when n is not divisible by p (RMB, 2003): for $p \nmid n$,

$$\Phi^n = \mu(n)\psi^n_S = \mu(n)\psi^n_\Lambda.$$

Factorisation (RMB and M. Schocker, 2007): if $n = kp^d$ where $p \nmid k$ then

$$\Phi^n = \Phi^{p^d} \circ \Phi^k.$$

Periodicity of Adams operations

Theorem. Suppose that G is finite. Then ψ^n_Λ is periodic in n if and only if the Sylow p-subgroups of G are cyclic.

The proof is fairly elementary, relying on the facts that if the Sylow p-subgroups are cyclic then there are only finitely many indecomposables (Higman) and the Green ring is semi-simple (Green and O'Reilly).
There is also a corresponding result for ψ^n_S.

Theorem. Suppose that G is finite. Then ψ^n_S is periodic in n if and only if the Sylow p-subgroups of G are cyclic.

The proof of this is more difficult. It relies on deep work of Symonds (2007), based on previous work of Karagueuzian and Symonds.

Cyclic p-groups

From now on we shall always assume that K has prime characteristic p and $G = C_{p^m}$. Recall that R_{KG} has $\mathbb{Z}$-basis $\{V_1, V_2, \ldots, V_{p^m}\}$.
What are $\psi^n_S(V_r)$ and $\psi^n_\Lambda(V_r)$?
We start with the case where $p \nmid n$ and write $\psi^n = \psi^n_S = \psi^n_\Lambda$.

Theorem. Let $G = C_{p^m}$. Suppose that $p \nmid n$. Let $r \in \{1, \ldots, p^m\}$. Take i such that $p^i < r \leqslant p^{i+1}$ and write $r = kp^i + s$ where $1 \leqslant s \leqslant p^i$ and $1 \leqslant k \leqslant p-1$. Then there is a formula (involving only elementary arithmetic) giving $\psi^n(V_r)$ in terms of $\psi^n(V_s)$ and $\psi^n(V_{p^i-s})$.

(Here we take $V_0 = 0$ to cover the case where $p^i - s = 0$.) This theorem gives $\psi^n(V_r)$ recursively on r.

The proof uses and extends work of Almkvist & Fossum, Kouwenhoven, Hughes & Kemper, and Gow & Laffey.

Patterns for cyclic p-groups

When we calculated ψ^n using Theorem 1 we noticed some interesting patterns, which we were later able to prove.

Example. Let $G = C_{25}$ where $p = 5$.

$$\begin{array}{rcl}
\psi^3(V_1) & = & V_1 \\
\psi^3(V_2) & = & V_4 - V_2 \\
\psi^3(V_3) & = & V_5 - V_3 + V_1 \\
\psi^3(V_4) & = & V_4 \\
\psi^3(V_5) & = & V_5 \\
\psi^3(V_6) & = & V_{16} - V_{14} + V_4 \\
\psi^3(V_7) & = & V_{19} - V_{17} + V_{13} - V_{11} + V_5 - V_3 + V_1 \\
\psi^3(V_8) & = & V_{20} - V_{18} + V_{16} - V_{14} + V_{12} - V_{10} + V_4 - V_2 \\
\psi^3(V_9) & = & V_{19} - V_{11} + V_1 \\
\psi^3(V_{10}) & = & V_{20} - V_{10} \\
\psi^3(V_{11}) & = & V_{21} - V_{11} + V_1 \\
\psi^3(V_{12}) & = & V_{24} - V_{22} + V_{20} - V_{14} + V_{12} - V_{10} + V_4 - V_2 \\
\psi^3(V_{13}) & = & V_{25} - V_{23} + V_{21} - V_{15} + V_{13} - V_{11} + V_5 - V_3 + V_1
\end{array}$$

An aside on cyclic 2-groups

Proposition. For $p = 2$ and G a cyclic 2-group, ψ_S^n and ψ_Λ^n are equal to the identity function for all odd n.

We also have $\Phi^n = \mu(n)\psi_S^n = \mu(n)\psi_\Lambda^n$ and

$$L^n(V) = \frac{1}{n} \sum_{d|n} \Phi^d(V^{n/d}).$$

Hence

$$L^n(V) = \frac{1}{n} \sum_{d|n} \mu(d) V^{n/d}.$$

By Möbius inversion we get a curiosity:

Corollary. For char $K = 2$, G a cyclic 2-group, V a KG-module and n odd,

$$V^n = \sum_{d|n} d\, L^d(V).$$

Heller translates

As before take $G = C_{p^m}$.
Recall that for a KG-module V, $\Omega(V)$ is defined up to isomorphism as the kernel of any map $P(V) \twoheadrightarrow V$ where $P(V)$ is the projective cover of V. Hence

$$\Omega(V_r) = V_{p^m - r} \text{ for } r = 1, \ldots, p^m$$

with the convention that $V_0 = 0$. We extend Ω to a $\mathbb{Z}$-linear map $\Omega : R_{KG} \to R_{KG}$. Also we write Ω^n for the composite of Ω taken n times. It is easily seen that

$$\Omega^n(V) = \begin{cases} V + aV_{p^m} & \text{if } n \text{ is even,} \\ \Omega(V) + aV_{p^m} & \text{if } n \text{ is odd,} \end{cases}$$

where a is some integer.

Reduction of ψ_S^n to ψ_Λ^n

As before take $G = C_{p^m}$.
Peter Symonds (2007) gave a recursive way of finding $S^n(V_r)$ in terms of exterior powers. His result leads to a corresponding result for Adams operations which is somewhat easier to state.

Theorem. Suppose that $p^{m-1} \leqslant r \leqslant p^m$. Then, for all n,

$$\psi_S^n(V_r) = (-1)^{n-1}\Omega^n(\psi_\Lambda^n(V_{p^m-r})) + (n, p^m)V_{p^m/(n,p^m)} + cV_{p^m}$$

where the integer c may be calculated by a dimension count if $\psi_\Lambda^n(V_{p^m-r})$ is known and (n, p^m) denotes the gcd of n and p^m.

This is easily seen to give ψ_S^n in terms of ψ_Λ^n. (For $r < p^{m-1}$ the module V_r may be regarded as a module for a proper factor group of G.)

ψ_Λ^n for a cyclic 2-group

The determination of $\Lambda^n(V_r)$ and $\psi_\Lambda^n(V_r)$ for a cyclic p-group is still open in general. But, as Frank Himstedt has told us, he and Peter Symonds have discovered a way of evaluating $\Lambda^n(V_r)$ in the case $p = 2$. This leads to a description of ψ_Λ^n as follows.

For $G = C_{2^m}$ we have already seen that ψ_Λ^n is the identity map when n is odd. Also, if $n = k2^d$ where k is odd then $\psi_\Lambda^n = \psi_\Lambda^k \circ \psi_\Lambda^{2^d}$. Thus it remains to describe $\psi_\Lambda^{2^d}$ for $d \geqslant 1$.

Theorem. Let $G = C_{2^m}$. Let $r \in \{1, \ldots, 2^m\}$. Take i such that $2^i < r \leqslant 2^{i+1}$ and write $r = 2^i + s$ where $1 \leqslant s \leqslant 2^i$. Then

$$\psi_\Lambda^{2^d}(V_r) = 2\psi_\Lambda^{2^{d-1}}(V_s) + \psi_\Lambda^{2^d}(V_{2^i-s}),$$

for $d \geqslant 2$, while

$$\psi_\Lambda^2(V_r) = 2V_{2^{i+1}} - 2V_{2^{i+1}-s} + \psi_\Lambda^2(V_{2^i-s}).$$

The result for ψ_Λ^2 can be obtained from work of Gow and Laffey (2006).

A conjecture on Lie resolvents

As we have seen, if $n = kp^d$ where $p \nmid k$ then $\Phi^n = \Phi^{p^d} \circ \Phi^k$ and $\Phi^k = \mu(k)\psi_S^k = \mu(k)\psi_\Lambda^k$. Thus it is reasonable to focus attention on the Lie resolvents Φ^{p^d}.

Conjecture. Let K have prime characteristic p and let G be a cyclic p-group. Then $\Phi^{p^2} = \Phi^{p^3} = \cdots = 0$.

This is true for G of order p (because of work of RMB, Kovács and Stöhr) and for G of order 4. There is good computer evidence in the case of a cyclic 2-group. Indeed, when $p = 2$ the conjecture may be reformulated in terms of Lie powers: roughly speaking, it says that, for $d \geqslant 2$, $L^{2^d}(V_r)$ is the sum of indecomposables of the form V_{2^i}. Here is some computer evidence for L^4.

$L^4(V_2)$	$=$	V_1	$+$	V_2								
$L^4(V_3)$	$=$			V_2	$+$	$4V_4$						
$L^4(V_4)$	$=$			$2V_2$	$+$	$14V_4$						
$L^4(V_5)$	$=$			V_2	$+$	$5V_4$	$+$	$16V_8$				
$L^4(V_6)$	$=$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$38V_8$				
$L^4(V_7)$	$=$					$3V_4$	$+$	$72V_8$				
$L^4(V_8)$	$=$					$4V_4$	$+$	$124V_8$				
$L^4(V_9)$	$=$					$3V_4$	$+$	$73V_8$	$+$	$64V_{16}$		
$L^4(V_{10})$	$=$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$40V_8$	$+$	$134V_{16}$		
$L^4(V_{11})$	$=$			V_2	$+$	$5V_4$	$+$	$19V_8$	$+$	$216V_{16}$		
$L^4(V_{12})$	$=$			$2V_2$	$+$	$14V_4$	$+$	$4V_8$	$+$	$316V_{16}$		
$L^4(V_{13})$	$=$			V_2	$+$	$4V_4$	$+$	$5V_8$	$+$	$440V_{16}$		
$L^4(V_{14})$	$=$	V_1	$+$	V_2			$+$	$6V_8$	$+$	$594V_{16}$		
$L^4(V_{15})$	$=$							$7V_8$	$+$	$784V_{16}$		
$L^4(V_{16})$	$=$							$8V_8$	$+$	$1016V_{16}$		
$L^4(V_{17})$	$=$							$7V_8$	$+$	$785V_{16}$	$+$	$256V_{32}$
$L^4(V_{18})$	$=$	V_1	$+$	V_2			$+$	$6V_8$	$+$	$596V_{16}$	$+$	$518V_{32}$
$L^4(V_{19})$	$=$			V_2	$+$	$4V_4$	$+$	$5V_8$	$+$	$443V_{16}$	$+$	$792V_{32}$
$L^4(V_{20})$	$=$			$2V_2$	$+$	$14V_4$	$+$	$4V_8$	$+$	$320V_{16}$	$+$	$1084V_{32}$
$L^4(V_{21})$	$=$			V_2	$+$	$5V_4$	$+$	$19V_8$	$+$	$221V_{16}$	$+$	$1400V_{32}$
$L^4(V_{22})$	$=$	V_1	$+$	V_2	$+$	$2V_4$	$+$	$40V_8$	$+$	$140V_{16}$	$+$	$1746V_{32}$
$L^4(V_{23})$	$=$					$3V_4$	$+$	$73V_8$	$+$	$71V_{16}$	$+$	$2128V_{32}$
$L^4(V_{24})$	$=$					$4V_4$	$+$	$124V_8$	$+$	$8V_{16}$	$+$	$2552V_{32}$
$L^4(V_{25})$	$=$					$3V_4$	$+$	$72V_8$	$+$	$9V_{16}$	$+$	$3024V_{32}$