Periodicity of Adams operations on the Green ring of a finite group.

Joint work with Roger Bryant

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1. The Green ring

2. Symmetric powers and exterior powers

3. Adams operations

4. Periodicity results

5. Cyclic p-groups.

1. The Green ring

Let G be any group, K a field.
Consider f.d. right KG-modules.

The Green ring (or representation ring) R_{KG} is the set of all formal $\mathbb{Z}$-linear combinations of isomorphism classes of KG-modules, with multiplication coming from tensor product over K.

[If U and V are KG-modules then $U \otimes_K V$ is a KG-module with diagonal action $(u \otimes v)g = ug \otimes vg$.]

Consider K as a trivial KG-module of dimension 1. Then since $U \otimes_K K \cong K \otimes_K U \cong U$ for all KG-modules U, we see that K (considered as a KG-module upto isomorphism) is the identity element of R_{KG}.

Notation

KG-modules	Elements of R_{KG}
U, V	U, V
$U \oplus V$	$U + V$
$U \otimes_K V$	$U . V$
$V^{\otimes n}$	V^n
K	1

By the Krull-Schmidt theorem we see that R_{KG} has a $\mathbb{Z}$-basis consisting of the isomorphism classes of indecomposable KG-modules.

Later on we shall be interested in the case where G is a finite group and K is a field of prime characteristic p. In this case, by a theorem of Higman, we have that R_{KG} has finite $\mathbb{Z}$-basis if and only if the Sylow p-subgroups of G are cyclic.

Example

K a field of characteristic $p > 0$.

$C = C_{p^m}$, a cyclic p-group of order $p^m \geq 1$.

$C = \langle g : g^{p^m} = 1 \rangle$.

For $r = 1, \ldots, p^m$, $KC(g-1)^r$ is a submodule of the regular module KC.

Let $V_r = KC / KC(g-1)^r$.

Then V_r is an indecomposable KC-module of dimension r.

In fact, every indecomposable KC-module is isomorphic to one of $V_1, \ldots, V_{p^m}$ so that $\{V_1, \ldots, V_{p^m}\}$ is a $\mathbb{Z}$-basis of R_{KC}.

Elements of R_{KC} have the form $\sum_{r=1}^{p^m} \alpha_r V_r$ where $\alpha_r \in \mathbb{Z}$ for $r = 1, \ldots, p^m$.

[Notice that $V_1 \cong K$ as KC-modules so that $V_1 = 1$ in R_{KC}]

2. Symmetric powers and exterior powers.

Let V be a vector space over K with basis $\{x_1, \dots, x_r\}$.

Write

$S(V)$ = free associative commutative algebra on $x_1, \dots, x_r$. $= K[x_1, \dots, x_r]$ "symmetric algebra"

$\Lambda(V)$ = free associative algebra on $x_1, \dots, x_r$ subject to $x_i \wedge x_i = 0$ and $x_i \wedge x_j = -x_j \wedge x_i$. = "exterior algebra".

Then $S(V)$ and $\Lambda(V)$ decompose into homogeneous components:

$$S(V) = S^0(V) \oplus S^1(V) \oplus \cdots \oplus S^n(V) \oplus \cdots$$

$$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \cdots \oplus \Lambda^n(V) \oplus \cdots$$

where $S^n(V)$ has basis $\{x_{i_1} \cdots x_{i_n} : 1 \leq i_1 \leq \cdots \leq i_n \leq r\}$

and $\Lambda^n(V)$ has basis $\{x_{i_1} \wedge \cdots \wedge x_{i_n} : 1 \leq i_1 < \cdots < i_n \leq r\}$.

Thus

$$\underbrace{\dim S^n(V) = \binom{n+r-1}{n}}_{\text{this grows as } n \text{ increases}}$$

and

$$\underbrace{\dim \Lambda^n(V) = \binom{r}{n}}_{\text{this is equal to zero if } n>r.}$$

We say that

$S^n(V)$ is the n^{th} symmetric power of V,
$\Lambda^n(V)$ is the n^{th} exterior power of V.

Now suppose that V is a KG-module. Then $S^n(V)$ and $\Lambda^n(V)$ become KG-modules by linear substitutions.

Notice that $S^0(V) \cong \Lambda^0(V) \cong K$
and $S^1(V) \cong \Lambda^1(V) \cong V$
as KG-modules.

(giving $S^0(V) = \Lambda^0(V) = 1$ and $S^1(V) = \Lambda^1(V) = V$ in R_{KG})

$$\boxed{\text{General Problem: Determine } S^n(V) \text{ and } \Lambda^n(V) \text{ up to isomorphism (ie as elements of } R_{KG}.}$$

3. Adams operations

We may extend R_{KG} to a ring $\mathbb{Q}R_{KG} = \mathbb{Q} \otimes_{\mathbb{Z}} R_{KG}$.

Consider the power series ring $\mathbb{Q}R_{KG}[[t]]$. For every KG-module V define $\psi_S^n(V)$ and $\psi_\Lambda^n(V)$ in $\mathbb{Q}R_{KG}$ by

$$\psi_S^1(V)t + \frac{1}{2}\psi_S^2(V)t^2 + \frac{1}{3}\psi_S^3(V)t^3 + \cdots$$
$$= \log(1 + S^1(V)t + S^2(V)t^2 + \cdots)$$

and

$$\psi_\Lambda^1(V)t - \frac{1}{2}\psi_\Lambda^2(V)t^2 + \frac{1}{3}\psi_\Lambda^3(V)t^3 - \cdots$$
$$= \log(1 + \Lambda^1(V)t + \Lambda^2(V)t^2 + \cdots)$$

It turns out that $\psi_S^n(V), \psi_\Lambda^n(V) \in R_{KG}$ for all KG-modules V and all $n > 0$, and

$$\psi_S^n(U+V) = \psi_S^n(U) + \psi_S^n(V)$$
$$\psi_\Lambda^n(U+V) = \psi_\Lambda^n(U) + \psi_\Lambda^n(V)$$

in R_{KG}.

Thus ψ_S^n and ψ_Λ^n extend to give $\mathbb{Z}$-linear functions

$$\psi_S^n, \psi_\Lambda^n : R_{KG} \to R_{KG}$$

called the Adams operations on R_{KG}.

The Adams operations are named for John Frank Adams who defined the operations ψ_Λ^k on R_{KG} and subsequently on the Grothendieck ring of real vector bundles. Adams used the latter to calculate the maximum number of linearly independent vector fields on the unit sphere S^{n-1} in $\mathbb{R}^n$.

$\Psi_S^1(V), \ldots, \Psi_S^n(V)$ are polynomials in $S^1(V), \ldots, S^n(V)$ and vice versa.

Similarly,

$\Psi_\Lambda^1(V), \ldots, \Psi_\Lambda^n(V)$ are polynomials in $\Lambda^1(V), \ldots, \Lambda^n(V)$ and vice versa.

<u>Examples</u>

- $\Psi_S^1(V) = \Psi_\Lambda^1(V) = V \; (= S^1(V) = \Lambda^1(V))$

- $\Psi_S^2(V) = 2S^2(V) - V^2$

  $\Psi_\Lambda^2(V) = V^2 - 2\Lambda^2(V)$.

- $\Psi_S^3(V) = 3S^3(V) - 3VS^2(V) + V^3$

  $\Psi_S^3(V) = 3\Lambda^3(V) - 3V\Lambda^2(V) + V^3$

Thus, knowledge of symmetric and exterior powers is equivalent to knowlegde of Adams operations.

<u>Problem</u>: Determine $\Psi_S^n(V)$ and $\Psi_\Lambda^n(V)$ as elements of $R_K G$.

Example $C = C_{25}$ $p = 5$. Consider $n=3$.

$$\psi^3(V_1) = V_1$$
$$\psi^3(V_2) = V_4 - V_2$$
$$\psi^3(V_3) = V_5 - V_3 + V_1$$
$$\psi^3(V_4) = V_4$$
$$\psi^3(V_5) = V_5$$
$$\psi^3(V_6) = V_{16} - V_{14} + V_4$$
$$\psi^3(V_7) = V_{19} - V_{17} + V_{13} - V_{11} + V_5 - V_3 + V_1$$
$$\psi^3(V_8) = V_{20} - V_{18} + V_{16} - V_{14} + V_{12} - V_{10} + V_4 - V_2$$
$$\psi^3(V_9) = V_{19} - V_{11} + V_1$$
$$\psi^3(V_{10}) = V_{20} - V_{10}$$
$$\psi^3(V_{11}) = V_{21} - V_{11} + V_1$$
$$\psi^3(V_{12}) = V_{24} - V_{22} + V_{20} - V_{14} + V_{12} - V_{10} + V_4 - V_2$$
$$\psi^3(V_{13}) = V_{25} - V_{23} + V_{21} - V_{15} + V_{13} - V_{11} + V_5 - V_3 + V_1$$

Properties of the Adams operations

Let G be a group, k a field of characteristic $p \geq 0$.

1) If $p \nmid n$ then $\psi_S^n = \psi_\Lambda^n$ and this map is a ring endomorphism of R_kG, and a $\mathbb{C}$-algebra endomorphism of $\mathbb{C} \otimes_{\mathbb{Z}} R_kG$. Also, $\psi^n(V)$ is a $\mathbb{Z}$-linear combination of direct summands of V^n.

2) If $p \nmid k$ then

$$\psi_S^{kn} = \psi_S^k \circ \psi_S^n \text{ and } \psi_\Lambda^{kn} = \psi_\Lambda^k \circ \psi_\Lambda^n$$

for all $n > 0$.

Now let G be a finite group, $G_{p'}$ the set of all p'-elements of G and Δ the $\mathbb{C}$-space of all functions $G_{p'} \to \mathbb{C}$ that are constant on G-conjugacy classes.

For any kG-module V, $\underbrace{Br(V)}_{\text{Brauer character of } V} \in \Delta$

We may extend linearly to obtain

$$Br : \mathbb{C} \otimes_{\mathbb{Z}} R_{kG} \to \Delta$$

③ $Br(\Psi_S^n(V))(g) = Br(\Psi_\Lambda^n(V))(g) = Br(V)(g^n)$

Brauer characters

roots of unity in $k \xrightarrow[\alpha]{\text{bijection}}$ Complex roots of unity of order coprime to p

$Br(V)(g) = \sum \alpha(\xi_i)$ ← eigenvalues of g acting on V.

(g: an elt. of $G_{p'}$.)

$Br(U) = Br(V) \iff U$ and V have the same composite factors

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4. Periodicity results

Proposition (essentially well-known).
Let G be a finite p'-group. Then

$\psi_S^n = \psi_\Lambda^n$ for all n and

$\psi^n = \psi^{n+\pi}$ where π is the exponent of G

Proof " This follows from (3) and the fact that kG is semisimple. $\square$

From now on let G be a finite group, k a field of prime characteristic p such that $p \mid |G|$.

Question: When are ψ_S^n and ψ_Λ^n periodic in n?

Theorem 1

Ψ_1^n is periodic in n $\iff$ the Sylow p-subgroups of G are cyclic.

Sketch of proof

$\Leftarrow$) Suppose that G has cyclic Sylow p-subgroups.

Then, as noted earlier, $R_K G$ is finite dimensional over $\mathbb{Z}$.

By a result of Green and O'Reilly, $\mathbb{C} \otimes R_K G$ is a semisimple algebra so that

$$\mathbb{C} \otimes R_K G = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_k$$

where $e_i^2 = e_i$, $e_i \cdot e_j = 0$ for $i \neq j$.

It follows that $\mathbb{C} \otimes R_K G$ has only finitely many idempotents and hence only finitely many algebra endomorphisms.

By ① $\{\Psi_1^n : n>0 \text{ and } p \nmid n\}$ is finite

Since there are finitely many indecomposables we may choose d such that $p^d > \dim V$ for every indecomposable V.

Let $n>0$ such that $p^d \nmid n$. Then we may write $n = p^i . k$ where $0 \le i < d$ and $p \nmid k$.

By ② we have that $\Psi_1^n = \Psi_1^k \circ \Psi_1^{p^i}$

Thus $\{\Psi_1^n : n>0 \text{ and } p^d \nmid n\}$ is finite. ⊛

Now fix an indecomposable V of dimension r. Consider r-tuples of the form

$$\Psi_c = (\Psi_1^c(V), \ldots, \Psi_1^{c+r-1}(V))$$

where $p^d \nmid c, \ldots, c+r-1$.

By ⊛ we have that $\Psi_a = \Psi_{a+s}$ for some $a, s \in \mathbb{N}$.

By the definition of $\Psi_\Lambda^n(V)$ and the fact that $\Lambda^n(V) = 0$ for $n > r$ we obtain Newton's formula

$$\Psi_\Lambda^n(V) - \Psi_\Lambda^{n-1}(V)\Lambda^1(V) + \cdots + (-1)^r \Psi_\Lambda^{n-r}(V)\Lambda^r(V) = 0$$

in $R_K G$.

Using this it is then easy to show that

$$\Psi_n = \Psi_{n+s} \iff \Psi_{n+1} = \Psi_{n+1+s}$$

Hence $\Psi_\Lambda^n(V) = \Psi_\Lambda^{n+s}(V)$ for all $n > 0$. Repeat this for each indecomposable module V and take the least common multiple of the periods.

($\Rightarrow$) Now suppose that the Sylow p-subgroups of G are <u>not</u> cyclic.

We show that the maps ψ_1^n are distinct for $p \nmid n$.

Main idea: Restrict to a minimal non-cyclic p-subgroup H of G.

Hence $H \cong C_p \times C_p$ or $p = 2$ and $H \cong \underbrace{Q_8}_{\text{quaternion group}}$.

i) <u>$H \cong C_p \times C_p$.</u> The Heller translates

$\Omega^n(K)$ are distinct non-projective indecomposable (↑ trivial KH-module)

for $n \geq 1$. Let $V = \Omega(K)$.

Then

$$V^{\otimes n} \cong \Omega^n(K) \oplus \text{projectives}.$$

By 1 we have that for $p \nmid n$

$\psi_1^n(V)$ is a $\mathbb{Z}$-linear combination of direct summands of $V^{\otimes n}$

Thus by consideration of dimensions we find that $\Omega^n(K)$ occurs in $\Psi_1^n(V)$ and hence the elements $\Psi_1^n(V)$ are distinct for $p \nmid n$.

ii) $\underline{p=2 \quad H \cong Q_8}$

More fiddly, but can be done!

$\square$

Theorem 2

For the cyclic group C_{p^m}, K a field of characteristic p,

$$\Psi_1^n = \Psi_1^{n+2p^m} \quad \text{for all } n>0$$

- Need quite detailed information about the Ψ_1^n for $p \nmid n$ to prove this.
- Proof also makes use of Newton's formula and uses the fact that $\Lambda^j(V) = 0$ for $j > \dim_K V$.

Theorems 1 and 2 also hold with Ψ_S^n in place of Ψ_Λ^n.

However, the proofs of these results are much more difficult, largely due to the fact that the $S^n(V)$ do not become zero for large n.

Main ideas

* Notice that the proof of Theorem 1 ($\Rightarrow$) also holds for Ψ_S^n.
* So it is enough to prove ($\Leftarrow$).

  [ie If the Sylow p-subgroups of G are cyclic then Ψ_S^n is periodic in n]
* By Conlon's induction theorem it is enough to show that the Ψ_S^n are periodic in n when G is an extension of a cyclic p-group by a cyclic p'-group.

* The key ingredient is then a result of Peter Symonds, which roughly says that (in this nice situation) the symmetric powers are periodic, modulo modules which are projective relative to proper cyclic p-subgroups of G.

* For Theorem 2, our knowledge of Ψ_S^n for $p \nmid n$ gives an exact period of $2p^m$ if p is odd and p^m if $p = 2$.

5. Cyclic p-groups

Let $C = C_{p^m}$, K a field of characteristi
$p > 0$.

Recall R_{KC} has $\mathbb{Z}$-basis $\{V_1, \dots, V_{p^m}\}$

Consider $\psi_S^n, \psi_\Lambda^n : R_{KC} \to R_{KC}$.

We have seen that

$$\left.\begin{aligned} \psi_\Lambda^n &= \psi_\Lambda^{n+2p^m} \\ \text{and } \psi_S^n &= \psi_S^{n+2p^m} \end{aligned}\right\} \text{ for all } n > 0$$

Moreover, ψ^n is a ring endomorphism of R_{KC} for $p \nmid n$. In fact, for $p \nmid n$, we may calculate $\psi^n(V_r)$ recursively in terms of $\psi^n(V_s)$ where $s < r$.

<u>Theorem 3</u> Let $n > 0$, $p \nmid n$ and let $1 \leq r \leq p^m$. Write $\lambda(r)$ for the smallest non-negative integer such that $r \leq p^{\lambda(r)}$.

(i) $\exists$ integers $j_1, \dots, j_l$ such that

$$p^{\lambda(r)} \geq j_1 > j_2 > \dots > j_l \geq 1$$

and $\psi^n(V_r) = V_{j_1} - V_{j_2} + V_{j_3} - \dots \pm V_{j_l}$

(ii) n even $\Rightarrow$ $j_1, \dots, j_l$ odd
n odd $\Rightarrow$ $j_1, \dots, j_l$ have same parity as r