Periodicity of Adams operations on the Green ring of a finite group

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Shameless self-promotion

Joint work with Professor Roger Bryant.

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'Periodicity of Adams operations on the Green ring of a finite group', Journal of Pure and Applied Algebra, (to appear).

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(You might also like to try:

'Adams operations on the Green ring of a cyclic group of prime-power order'

in Journal of Algebra, 323 (2010).)

The Green ring

Let K be a field of prime characteristic p and let G be a finite group. We consider finite-dimensional right KG-modules.

The **Green ring** (or representation ring) R_{KG} has $\mathbb{Z}$-basis consisting of the isomorphism classes of (f. d.) indecomposable KG-modules with multiplication coming from tensor product.

KG-modules:	$U \oplus V$	$U \otimes_K V$	$V^{\otimes n}$
Elements of R_{KG} :	$U + V$	UV	V^n

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Example. Let $G = \langle g \rangle$ be a cyclic p-group of order q. There are q indecomposable KG-modules up to isomorphism.

For $r = 1, \ldots, q$ write $V_r = KG/KG(g-1)^r$.
Then V_r is indecomposable of dimension r and hence R_{KG} has $\mathbb{Z}$-basis $\{V_1, \ldots, V_q\}$.

Symmetric and exterior powers

Let V be a vector space over K with basis $\{x_1, \ldots, x_r\}$. Write

$S(V) = K[x_1, \ldots, x_r]$ (free associative commutative K-algebra),
$\Lambda(V) =$ free associative K-algebra on $x_1, \ldots, x_r$ subject to
$x_i \wedge x_i = 0$ and $x_i \wedge x_j = -x_j \wedge x_i$.

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Take decompositions into homogeneous components:
$S(V) = S^0(V) \oplus S^1(V) \oplus \cdots \oplus S^n(V) \oplus \cdots,$
$\Lambda(V) = \Lambda^0(V) \oplus \Lambda^1(V) \oplus \cdots \oplus \Lambda^n(V) \oplus \cdots$

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If V is a KG-module then $S^n(V)$, and $\Lambda^n(V)$ become KG-modules by linear substitutions.

$S^0(V) \cong \Lambda^0(V) \cong K$, written as 1 in R_{KG}.
$S^1(V) \cong \Lambda^1(V) \cong V$.

Adams operations

Consider the power series ring $(\mathbb{Q} \otimes R_{KG})[[t]]$. Define $\psi^n_S(V)$ and $\psi^n_\Lambda(V)$ in $\mathbb{Q} \otimes R_{KG}$ by

$$\begin{aligned}\psi^1_S(V)t + \tfrac{1}{2}\psi^2_S(V)t^2 + \tfrac{1}{3}\psi^3_S(V)t^3 + \cdots \\ = \log(1 + S^1(V)t + S^2(V)t^2 + \cdots),\end{aligned}$$

$$\begin{aligned}\psi^1_\Lambda(V)t - \tfrac{1}{2}\psi^2_\Lambda(V)t^2 + \tfrac{1}{3}\psi^3_\Lambda(V)t^3 - \cdots \\ = \log(1 + \Lambda^1(V)t + \Lambda^2(V)t^2 + \cdots).\end{aligned}$$

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$$\psi^1_\Lambda(V)t - \tfrac{1}{2}\psi^2_\Lambda(V)t^2 + \tfrac{1}{3}\psi^3_\Lambda(V)t^3 - \cdots = \log(1 + \Lambda^1(V)t + \Lambda^2(V)t^2 + \cdots).$$

It turns out that $\psi^n_S(V), \psi^n_\Lambda(V) \in R_{KG}$ and

$$\psi^n_S(U + V) = \psi^n_S(U) + \psi^n_S(V), \;\; \psi^n_\Lambda(U + V) = \psi^n_\Lambda(U) + \psi^n_\Lambda(V).$$

Thus we get $\mathbb{Z}$-linear functions called the **Adams operations**:

$$\psi^n_S, \psi^n_\Lambda : R_{KG} \to R_{KG}.$$

Properties of Adams operations

The main properties of the Adams operations on R_{KG} were given by Benson (1984) and RMB (2003) following ideas of Adams, Frobenius and others.

Linearity.
As we have seen, ψ_S^n and ψ_Λ^n are $\mathbb{Z}$-linear maps.

'Nice' behaviour when n is not divisible by p.
For $p \nmid n$, $\psi_S^n = \psi_\Lambda^n$, and ψ_S^n is a ring endomorphism of R_{KG}.

Factorisation property.
If $n = kp^d$ where $p \nmid k$ then

$$\psi_S^n = \psi_S^k \circ \psi_S^{p^d}, \quad \psi_\Lambda^n = \psi_\Lambda^k \circ \psi_\Lambda^{p^d}.$$

Periodicity of Adams operations

Theorem 1. *ψ^n_Λ is periodic in n if and only if the Sylow p-subgroups of G are cyclic.*

The proof is fairly elementary, relying on the facts that if the Sylow p-subgroups are cyclic then there are only finitely many indecomposables (Higman) and the Green ring is semi-simple (Green and O'Reilly).

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There is also a corresponding result for ψ^n_S.

Theorem 2. *ψ^n_S is periodic in n if and only if the Sylow p-subgroups of G are cyclic.*

The proof of this is more difficult. It relies on deep work of Symonds (2007), based on previous work of Karagueuzian and Symonds.

Finally...

ごせいちょう　ありがとう　ございました

(Thank you for your attention.)

Example: Cyclic p-groups

Theorem 3. *Let G be a cyclic p-group of order $q > 1$. Then*

(i) $\psi_\Lambda^n = \psi_\Lambda^{n+2q}$ *for all* $n > 0$.

(ii) $\psi_S^n = \psi_S^{n+2q}$ *for all* $n > 0$.

Note: If $\psi_\Lambda^n = \psi_\Lambda^{n+m}$ for all $n > 0$ then $2q \mid m$, i.e. this is the minimum period for ψ_Λ^n.
The minimum period for ψ_S^n is $2q$ if p is odd and q if p is even.