Adams operations on the Green ring of a finite group

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Shameless self-promotion

Joint work with Professor Roger Bryant.

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'Periodicity of Adams operations on the Green ring of a finite group', Journal of Pure and Applied Algebra, 215 (2011), 989–1002.

'Adams operations on the Green ring of a cyclic group of prime-power order' Journal of Algebra, 323 (2010), 2818–2833.

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The **Green ring** (or representation ring) R_{KG} has $\mathbb{Z}$-basis consisting of the isomorphism classes of (f. d.) indecomposable KG-modules with multiplication coming from tensor product.

$$\begin{array}{rccc} KG\text{-modules:} & U \oplus V & U \otimes_K V & V^{\otimes n} \\ \text{Elements of } R_{KG}: & U + V & UV & V^n \end{array}$$

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Notice that the one-dimensional module on which G acts trivially is the identity element in R_{KG}. Thus $K = 1$ in R_{KG}.

The Green ring of a cyclic p-group

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Then V_r is indecomposable of dimension r and hence R_{KG} has $\mathbb{Z}$-basis $\{V_1, \ldots, V_q\}$.

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Each indecomposable V_r has basis $\{y_1, \ldots, y_r\}$ and the action of g on V_r with respect to this basis is given by the Jordan block

$$\begin{pmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}.$$

(Notice that V_1 is the one-dimensional trivial module and V_q is the regular KC-module.)

Symmetric and exterior powers

Let V be a vector space over K with basis $\{x_1, \ldots, x_r\}$. Write

$S(V) = K[x_1, \ldots, x_r]$ (free associative commutative K-algebra),
$\Lambda(V) =$ free associative K-algebra on $x_1, \ldots, x_r$ subject to

$$x_i \wedge x_i = 0 \text{ and } x_i \wedge x_j = -x_j \wedge x_i.$$

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Take decompositions into homogeneous components:
$S(V) = S^0(V) \oplus S^1(V) \oplus \cdots \oplus S^n(V) \oplus \cdots,$
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These components are the **symmetric powers** and **exterior powers** of V.

If V is a KG module then $S^n(V)$ and $\Lambda^n(V)$ become KG-modules by linear substitutions.

Properties of symmetric and exterior powers

The nth symmetric power $S^n(V)$ has K-basis

$$\{x_{i_1}x_{i_2}\cdots x_{i_n} : 1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \leqslant r\}.$$

The nth exterior power $\Lambda^n(V)$ has K-basis

$$\{x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_n} : 1 \leqslant i_1 < i_2 < \cdots < i_n \leqslant r\}.$$

Thus $\dim S^n(V) = \binom{n+r-1}{n}$ and $\dim \Lambda^n(V) = \binom{r}{n}$.

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It is also easy to check that

$$\begin{aligned} S^n(U \oplus V) &\cong \bigoplus_{a+b=n} S^a(U) \otimes S^b(V) \\ \text{and } \Lambda^n(U \oplus V) &\cong \bigoplus_{a+b=n} \Lambda^a(U) \otimes \Lambda^b(V). \end{aligned}$$

Motivating problem

We started with a finite-dimensional KG-module V and have created two families of KG-modules. What can we say about these new modules?

Problem. Determine $S^n(V)$ and $\Lambda^n(V)$ up to isomorphism, i.e. as elements of R_{KG}.

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Examples.
$S^0(V) \cong \Lambda^0(V) \cong K$, written as 1 in R_{KG}.
$S^1(V) \cong \Lambda^1(V) \cong V$.

Note that $S^n(V) \ncong \Lambda^n(V)$ for $n > 1$, by dimensions.
In particular, $\Lambda^n(V) = 0$ for $n > r$, whilst $S^n(V) \neq 0$ for all n.

Adams operations

Consider the power series ring $(\mathbb{Q} \otimes R_{KG})[[t]]$.
Define $\psi_S^n(V)$ and $\psi_\Lambda^n(V)$ in $\mathbb{Q} \otimes R_{KG}$ by

$$\psi_S^1(V)t + \tfrac{1}{2}\psi_S^2(V)t^2 + \tfrac{1}{3}\psi_S^3(V)t^3 + \cdots = \log(1 + S^1(V)t + S^2(V)t^2 + \cdots),$$

$$\psi_\Lambda^1(V)t - \tfrac{1}{2}\psi_\Lambda^2(V)t^2 + \tfrac{1}{3}\psi_\Lambda^3(V)t^3 - \cdots = \log(1 + \Lambda^1(V)t + \Lambda^2(V)t^2 + \cdots).$$

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It turns out that $\psi_S^n(V), \psi_\Lambda^n(V) \in R_{KG}$ and

$$\psi_S^n(U+V) = \psi_S^n(U) + \psi_S^n(V), \quad \psi_\Lambda^n(U+V) = \psi_\Lambda^n(U) + \psi_\Lambda^n(V).$$

Thus we get $\mathbb{Z}$-linear functions called the **Adams operations**:

$$\psi_S^n, \psi_\Lambda^n : R_{KG} \to R_{KG}.$$

Adams operations

Clearly $\psi^1_S(V), \ldots, \psi^n_S(V)$ are polynomials in $S^1(V), \ldots, S^n(V)$ and vice versa. Similarly for the exterior powers.

Thus knowledge of the symmetric and exterior powers in R_{KG} is **equivalent** to knowledge of the Adams operations (assuming we know how to multiply in R_{KG}).

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For now it is perhaps best to think of Adams operations as providing an attractive re-packaging of results on exterior and symmetric powers rather than a tool for proving theorems about these modules.

Properties of Adams operations

The main properties of the Adams operations on R_{KG} were given by Benson (1984) and RMB (2003) following ideas of Adams, Frobenius and others.

Linearity.
As we have seen, ψ^n_S and ψ^n_Λ are $\mathbb{Z}$-linear maps.

'Nice' behaviour when n is not divisible by p.
For $p \nmid n$, $\psi^n_S = \psi^n_\Lambda$, and ψ^n_S is a ring endomorphism of R_{KG}.

Factorisation property.
If $n = kp^d$ where $p \nmid k$ then

$$\psi^n_S = \psi^k_S \circ \psi^{p^d}_S, \quad \psi^n_\Lambda = \psi^k_\Lambda \circ \psi^{p^d}_\Lambda .$$

Periodicity of Adams operations

Theorem 1. *ψ_Λ^n is periodic in n if and only if the Sylow p-subgroups of G are cyclic.*

The proof is fairly elementary, relying on the facts that if the Sylow p-subgroups are cyclic then there are only finitely many indecomposables (Higman) and the Green ring is semi-simple (Green and O'Reilly).

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There is also a corresponding result for ψ^n_S.

Theorem 2. *ψ^n_S is periodic in n if and only if the Sylow p-subgroups of G are cyclic.*

The proof of this is more difficult. It relies on deep work of Symonds (2007), based on previous work of Karagueuzian and Symonds.

Minimum period

Suppose now that the Sylow p-subgroups of G are cyclic. Thus ψ_S^n and ψ_Λ^n are both periodic in n and we would like to calculate the minimum periods. Let e denote the exponent of G.

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(i) ψ_S^n is periodic in n with minimum period $\mathrm{lcm}(2, \mathrm{e})$;

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(ii) ψ_Λ^n is periodic in n with minimum period $2e$.

G has proper cyclic Sylow p-subgroup.
We obtain a lower bound; ψ_S^n and ψ_Λ^n are periodic in n with minimum period divisible by $\mathrm{lcm}(2, e)$.

Cyclic p-groups

Let G be a cyclic p-group of order $q > 1$.
Recall that R_{KG} has $\mathbb{Z}$-basis $\{V_1, V_2, \ldots, V_q\}$.

What are $\psi^n_S(V_r)$ and $\psi^n_\Lambda(V_r)$?
We start with the case where $p \nmid n$ and write $\psi^n = \psi^n_S = \psi^n_\Lambda$.

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Theorem 3. *Suppose that $p \nmid n$ and let $r \in \{1, \ldots, q\}$.
Write $r = kp^i + s$ where $1 \leqslant k \leqslant p-1$ and $1 \leqslant s \leqslant p^i$.
Then there is a formula (involving only elementary arithmetic) giving $\psi^n(V_r)$ in terms of $\psi^n(V_s)$ and $\psi^n(V_{p^i-s})$.*

(Here we take $V_0 = 0$ to cover the case where $p^i - s = 0$.)
This theorem gives $\psi^n(V_r)$ recursively on r.

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(Here we take $V_0 = 0$ to cover the case where $p^i - s = 0$.)
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The proof uses and extends work of Almkvist & Fossum, Kouwenhoven, Hughes & Kemper, and Gow & Laffey.

Patterns for cyclic p-groups

When we calculated ψ^n using **Theorem 3** we noticed some interesting patterns, which we were later able to prove.

Example. Let $G = C_{25}$ where $p = 5$.

$$\begin{aligned}
\psi^3(V_1) &= V_1 \\
\psi^3(V_2) &= V_4 - V_2 \\
\psi^3(V_3) &= V_5 - V_3 + V_1 \\
\psi^3(V_4) &= V_4 \\
\psi^3(V_5) &= V_5 \\
\psi^3(V_6) &= V_{16} - V_{14} + V_4 \\
\psi^3(V_7) &= V_{19} - V_{17} + V_{13} - V_{11} + V_5 - V_3 + V_1 \\
\psi^3(V_8) &= V_{20} - V_{18} + V_{16} - V_{14} + V_{12} - V_{10} + V_4 - V_2 \\
\psi^3(V_9) &= V_{19} - V_{11} + V_1 \\
\psi^3(V_{10}) &= V_{20} - V_{10} \\
\psi^3(V_{11}) &= V_{21} - V_{11} + V_1 \\
\psi^3(V_{12}) &= V_{24} - V_{22} + V_{20} - V_{14} + V_{12} - V_{10} + V_4 - V_2 \\
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\end{aligned}$$

Patterns for cyclic p-groups

When we calculated ψ^n using **Theorem 4** we noticed some interesting patterns, which we were later able to prove.

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Heller translates

Recall that for a KG-module V, $\Omega(V)$ is defined up to isomorphism as the kernel of any map $P(V) \twoheadrightarrow V$ where $P(V)$ is the projective cover of V.

Hence

$$\Omega(V_r) = V_{q-r} \quad \text{for } r = 1, \ldots, q$$

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with the convention that $V_0 = 0$.

We extend Ω to a $\mathbb{Z}$-linear map $\Omega : R_{KG} \to R_{KG}$. Also we write Ω^n for the composite of Ω taken n times. It is easily seen that

$$\Omega^n(V) = \begin{cases} V + aV_q & \text{if } n \text{ is even,} \\ \Omega(V) + aV_q & \text{if } n \text{ is odd,} \end{cases}$$

where a is some integer.

Reduction of ψ_S^n to ψ_Λ^n

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Theorem 4. *Suppose that $q/p \leqslant r \leqslant q$. Then, for all n,*

$$\psi_S^n(V_r) = (-1)^{n-1}\Omega^n(\psi_\Lambda^n(V_{q-r})) + (n,q)V_{q/(n,q)} + cV_q$$

where the integer c may be calculated by a dimension count if $\psi_\Lambda^n(V_{q-r})$ is known and (n,q) denotes the gcd of n and q.

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This is easily seen to give ψ_S^n in terms of ψ_Λ^n.
(For $r < q/p$ the module V_r may be regarded as a module for a proper factor group of G.)

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  We gave a lower bound for the minimum periods.
- When G is a cyclic p-group we gave recursive formula to calculate $\psi_S^n = \psi_\Lambda^n$ for n not divisible by p.
  This recursion gives rise to some nice patterns.
- For cyclic p-groups we also showed that $\psi_S^n(V_r)$ can be expressed in terms of $\psi_\Lambda^n(V_{q-r})$, where V_{q-r} is the Heller translate of V_r.

Cyclic 2-groups

The determination of $\Lambda^n(V_r)$ and $\psi^n_\Lambda(V_r)$ for a cyclic p-group is still open in general. Frank Himstedt and Peter Symonds have recently discovered a way of evaluating $\Lambda^n(V_r)$ in the case $p = 2$. This leads to a description of ψ^n_Λ as follows.

- It can be shown that ψ^n_Λ is equal to the identity function for all odd n.
- Also, if $n = k2^d$ where k is odd then $\psi^n_\Lambda = \psi^k_\Lambda \circ \psi^{2^d}_\Lambda$.
- Thus it remains to describe $\psi^{2^d}_\Lambda$ for $d \geqslant 1$.

Theorem 5. *Let G be a cyclic 2-group. Write $r = 2^i + s$ where $1 \leqslant s \leqslant 2^i$. Then*

$$\begin{aligned} \psi^2_\Lambda(V_r) &= 2V_{2^{i+1}} - 2V_{2^{i+1}-s} + \psi^2_\Lambda(V_{2^i-s}) \\ \textit{and}\ \ \psi^{2^d}_\Lambda(V_r) &= 2\psi^{2^{d-1}}_\Lambda(V_s) + \psi^{2^d}_\Lambda(V_{2^i-s})\ \ \textit{for}\ \ d \geqslant 2. \end{aligned}$$

(ψ^2_Λ can also be obtained from work of Gow and Laffey (2006)).