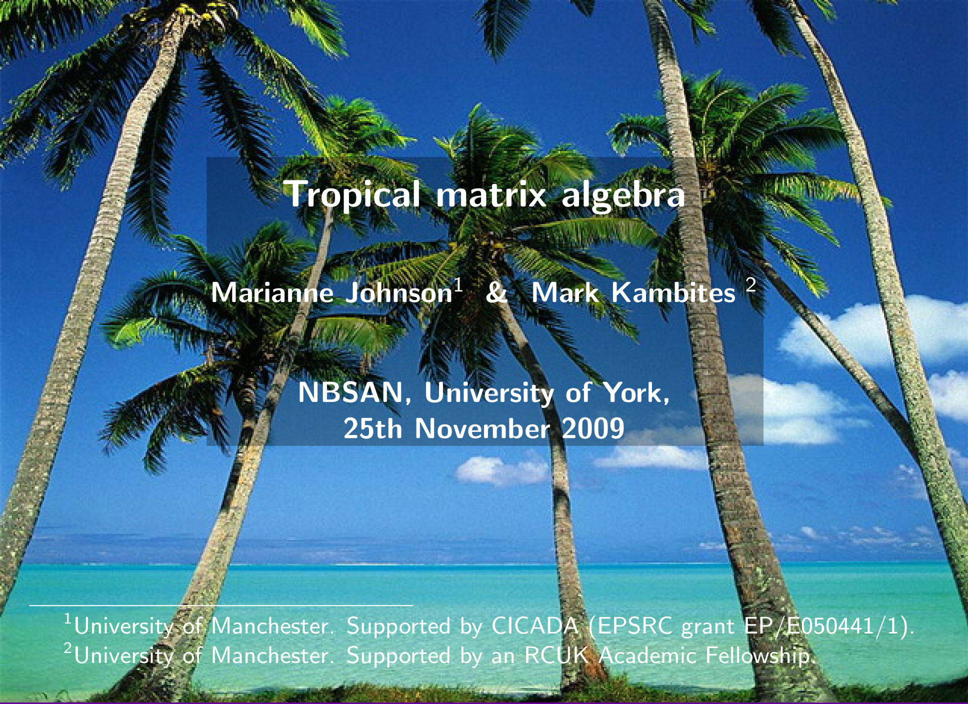

Tropical matrix algebra

Marianne Johnson[1] & Mark Kambites[2]

NBSAN, University of York,
25th November 2009

[1]University of Manchester. Supported by CICADA (EPSRC grant EP/E050441/1).
[2]University of Manchester. Supported by an RCUK Academic Fellowship.

The tropical semiring

The **tropical** (or **max-plus**) semiring has elements

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$$

and binary operations

- $x \oplus y = \max(x, y)$; and
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Properties

$\overline{\mathbb{R}}$ *is an* **idempotent semifield***:*

- $(\mathbb{R}, \otimes)$ *is an abelian group with identity* 0*;*
- $-\infty$ *is a zero element for* $\otimes$*;*
- $(\overline{\mathbb{R}}, \oplus)$ *is a commutative monoid with identity* $-\infty$*;*
- $\otimes$ *distributes over* $\oplus$*;*
- $x \oplus x = x$

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Tropical methods have applications in ...

- *Combinatorial Optimisation*
- *Discrete Event Systems*
- *Control Theory*
- *Formal Languages and Automata*
- *Phylogenetics*
- *Statistical Inference*
- *Geometric Group Theory*
- *Enumerative Algebraic Geometry*

Tropical matrices

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What is its abstract algebraic structure?

For example, what are its . . .

- *Ideals?*
- *Idempotents?*
- *Subgroups?*

Affine tropical n-space

$M_n(\overline{\mathbb{R}})$ comes equipped with a natural action on the space $\overline{\mathbb{R}}^n$ of **tropical** n-**vectors** (**affine tropical** n-**space**).

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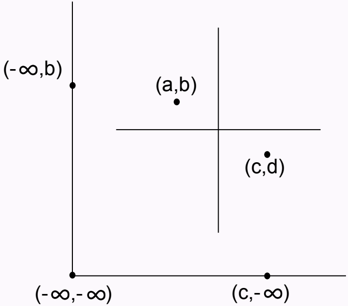

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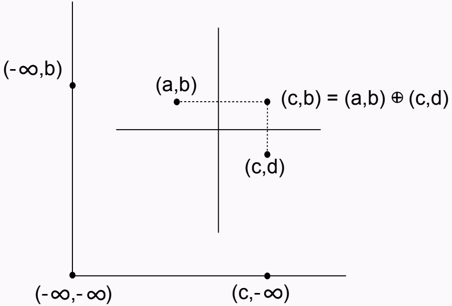

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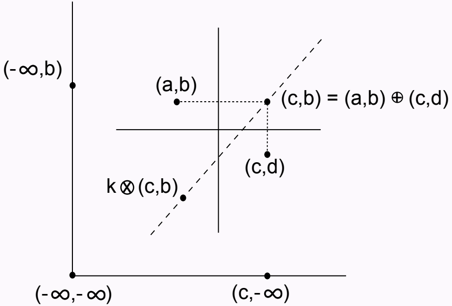

Projective tropical $(n-1)$-space

From $\overline{\mathbb{R}}^n$ we obtain **projective tropical** $(n-1)$-**space** by discarding the "zero vector" and identifying two vectors which are "tropical scalings" of each other.

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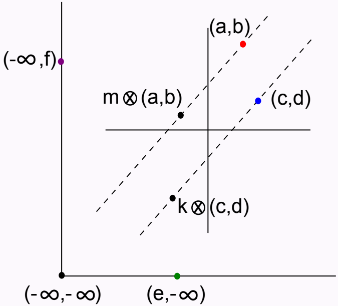

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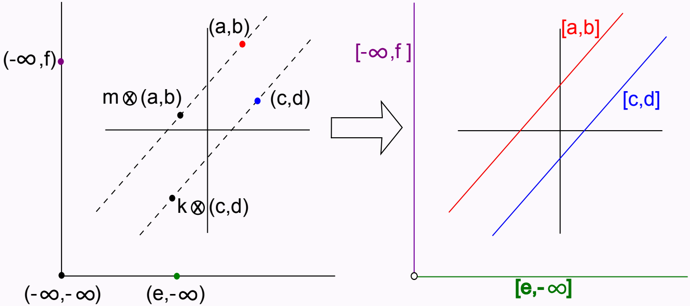

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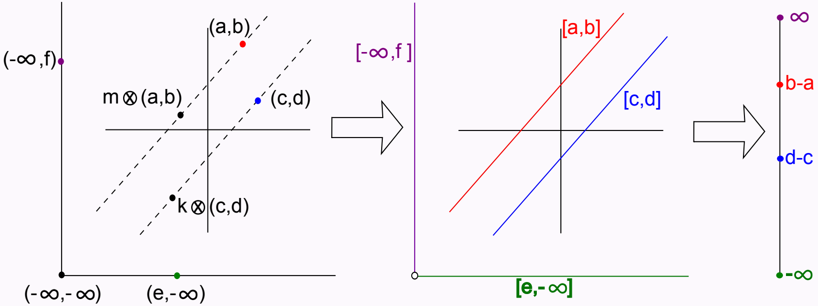
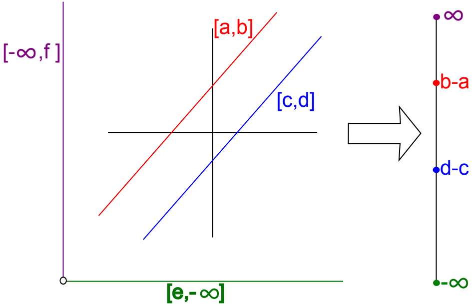


Projective tropical 1-space

Thus we identify projective tropical 1-space with the two-point compactification of the real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ via the map

$$[x, y] \mapsto y - x.$$



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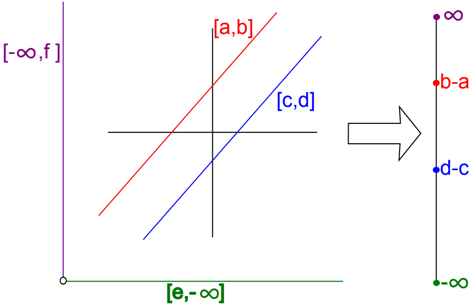

Question

How does the **algebraic** *structure of $M_n(\overline{\mathbb{R}})$ relate to the* **geometric** *structure of affine tropical n-space and projective tropical $(n-1)$-space?*

Column and row spaces

For $A \in M_n(\overline{\mathbb{R}})$ we write

- $C(A)$ for the column span of A (a tropical 'subspace' in $\overline{\mathbb{R}}^n$);
- $R(A)$ for the row span of A (a tropical 'subspace' in $\overline{\mathbb{R}}^n$).

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Let $A = \begin{pmatrix} a & -\infty \\ b & c \end{pmatrix} \in M_2(\overline{\mathbb{R}})$*, where* $a, b, c \in \mathbb{R}$*.*

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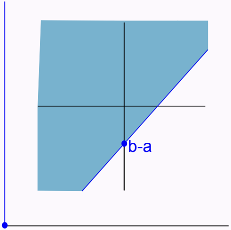
$$\text{Then } C(A) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x + b - a \leq y \right\} \subseteq \overline{\mathbb{R}}^2.$$



Projective column and row spaces

For $A \in M_n(\overline{\mathbb{R}})$ we write

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In the case $n = 2$, convex sets in $\hat{\mathbb{R}}$ are just intervals.

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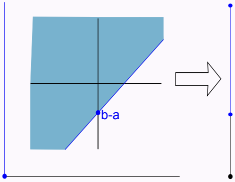

Then $PC(A) = [b - a, \infty] \subseteq \hat{\mathbb{R}}$.

Ideals and Green's relations

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Similarly ...

- $x \leq_{\mathcal{L}} y \iff Mx \subseteq My, \quad x \mathcal{L} y \iff Mx = My$
- $x \leq_{\mathcal{J}} y \iff MxM \subseteq MyM, \quad x \mathcal{J} y \iff MxM = MyM;$

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Note

These relations encapsulate the (left, right and two-sided) ideal structure of M and are fundamental to its structure.

Green's $\mathcal{R}$ relation in $M_n(\overline{\mathbb{R}})$.

Lemma

Let $A, B \in M_n(\overline{\mathbb{R}})$. Then the following are equivalent:

(i) $A \leq_{\mathcal{R}} B$;

(ii) $C(A) \subseteq C(B)$;

(iii) $PC(A) \subseteq PC(B)$.

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So $\mathcal{R}$-classes in $M_n(\overline{\mathbb{R}})$ are in 1-1 correspondence with n-generated convex sets in projective tropical $(n-1)$-space.

Green's $\mathcal{L}$ relation in $M_n(\overline{\mathbb{R}})$.

Lemma

Let $A, B \in M_n(\overline{\mathbb{R}})$. Then the following are equivalent:

(i) $A \leq_{\mathcal{L}} B$;

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Green's $\mathcal{R}$ relation in $M_2(\overline{\mathbb{R}})$.

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Corollary

The lattice of principal right ideals in $M_2(\overline{\mathbb{R}})$ is isomorphic to the intersection lattice generated by closed subintervals of the closed unit interval.

Isometries in projective tropical 1-space

We can define a "metric" on $\hat{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \infty & \text{if } x = -\infty \neq y \text{ or } x = \infty \neq y \\ |y - x| & \text{otherwise.} \end{cases}$$

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Proposition

Let $A \in M_2(\overline{\mathbb{R}})$. Then $PC(A) \cong PR(A)$.

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Let $A, B \in M_2(\overline{\mathbb{R}})$*. Then* $A \leq_{\mathcal{J}} B$ *if and only if* $PC(A)$ *embeds isometrically in* $PC(B)$*.*

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Corollary

The lattice of principal two-sided ideals in $M_2(\overline{\mathbb{R}})$ is isomorphic to the lattice of isometry types of closed convex subsets of $\hat{\mathbb{R}}$.

Idempotents and regularity

The idempotents in $M_2(\overline{\mathbb{R}})$ are

$$\begin{pmatrix} 0 & x \\ y & x+y \end{pmatrix}, \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}, \begin{pmatrix} x+y & x \\ y & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -\infty & -\infty \\ -\infty & -\infty \end{pmatrix}$$

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For every 2-generated convex subset X of $\hat{\mathbb{R}}$, there is an idempotent $E \in M_2(\overline{\mathbb{R}})$ with $PC(E) = X$. Thus $M_2(\overline{\mathbb{R}})$ is **regular**.

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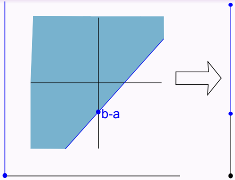
Example

Consider $X = [b - a, \infty] \subseteq \hat{\mathbb{R}}$.
Then we can choose

$$E = \begin{pmatrix} 0 & -\infty \\ b - a & 0 \end{pmatrix} \in M_2(\overline{\mathbb{R}})$$

such that $PC(E) = X$



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Theorem

Let $M \subseteq \hat{\mathbb{R}}$ be a closed convex subset. The maximal subgroups in the $\mathcal{D}$-class corresponding to M are isomorphic to:

- $\{1\}$ *if* $M = \emptyset$;
- $\mathbb{R}$ *if* M *is a point or an interval with one real endpoint;*
- $\mathbb{R} \times S_2$ *if* M *is an interval with 2 real endpoints;*
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Corollary

Every group of 2×2 *tropical matrices is torsion-free abelian, or has a torsion-free abelian subgroup of index* 2.