Modular Lie powers of relation modules and free central extensions of groups

Marianne Johnson
(joint work with Ralph Stöhr)

Groups at St Andrews, Bath, 2009

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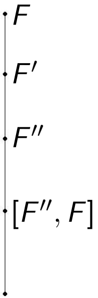

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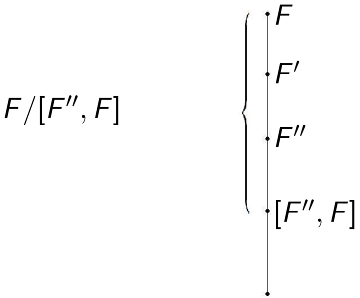

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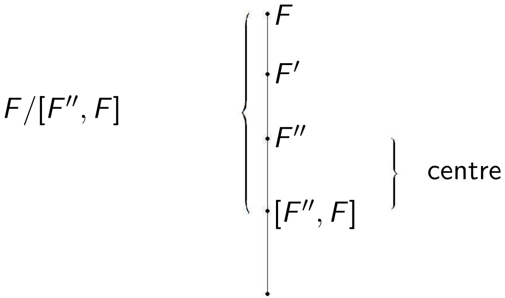

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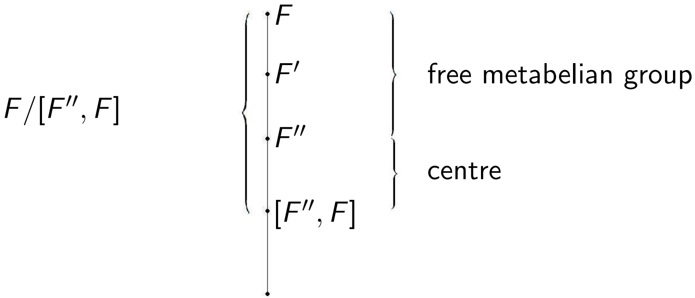

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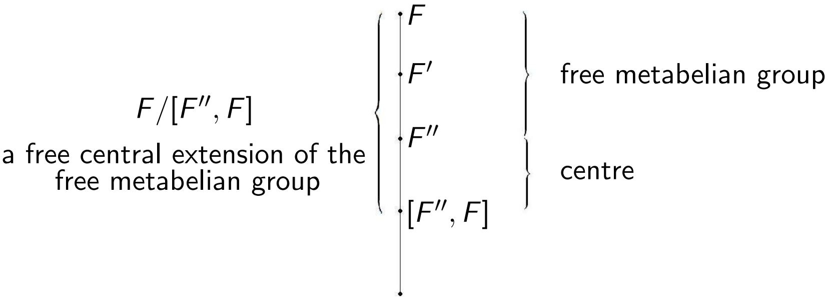

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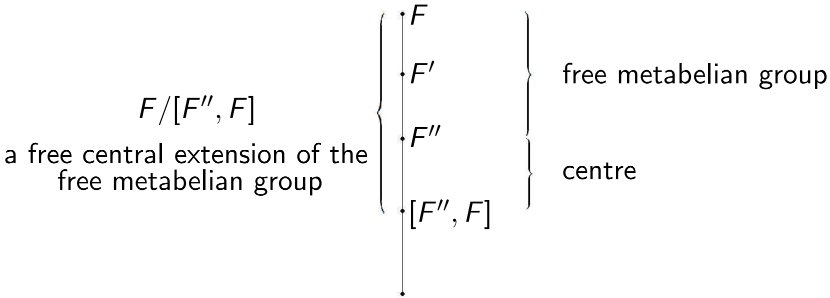

The free centre-by-metabelian group is the free group in the variety of groups determined by the identical relation

$$[[[x_1, x_2], [x_3, x_4]], y] \equiv 1.$$

The torsion subgroup of $F/[F'', F]$

Theorem (C.K. Gupta, 1973)

The free centre-by-metabelian group $F/[F'', F]$ of rank n is torsion free for $n = 2, 3$, and for $n \geq 4$ it contains an elementary abelian 2-group of rank $\binom{n}{4}$ in its centre.

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The torsion subgroup is generated by the elements

$$\begin{aligned} w(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}) = &[[x_{i_1}, x_{i_2}], [x_{i_3}^{-1}, x_{i_4}^{-1}]]\,[[x_{i_3}, x_{i_4}], [x_{i_1}^{-1}, x_{i_2}^{-1}]] \\ &[[x_{i_1}, x_{i_3}], [x_{i_4}^{-1}, x_{i_2}^{-1}]]\,[[x_{i_4}, x_{i_2}], [x_{i_1}^{-1}, x_{i_3}^{-1}]] \\ &[[x_{i_1}, x_{i_4}], [x_{i_2}^{-1}, x_{i_3}^{-1}]]\,[[x_{i_2}, x_{i_3}], [x_{i_1}^{-1}, x_{i_4}^{-1}]] \end{aligned}$$

where $x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4} \in X$ and $i_1 < i_2 < i_3 < i_4$.

Connection to homology groups

The Crucial Observation (Yu. V. Kuz'min, 1977)

Gupta's torsion subgroup of $F''/[F'', F]$ is isomorphic to the integral homology group $H_4(F/F')$ reduced modulo 2:

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Kuz'min linked the torsion in the free centre-by-metabelian group $F/[F'', F]$ to the fourth homology group $H_4(F/F')$ of the free abelian group F/F' and gave a new proof of Gupta's Theorem using homological methods.

Generalization

We may write the free centre-by-metabelian group as

$$F/[F'', F] = F/[(F')', F] = F/[\gamma_2(F'), F]$$

where $\gamma_2(F')$ is the second term of the lower central series of F'.

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For the purposes of this talk, we shall restrict attention to the case $R = F'$.

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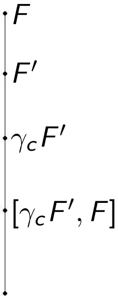

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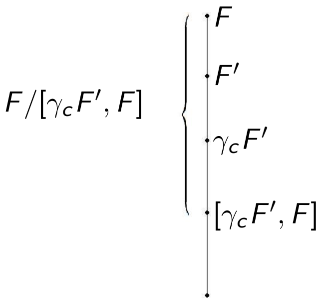

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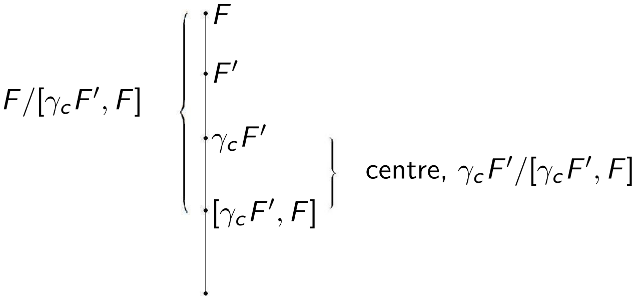

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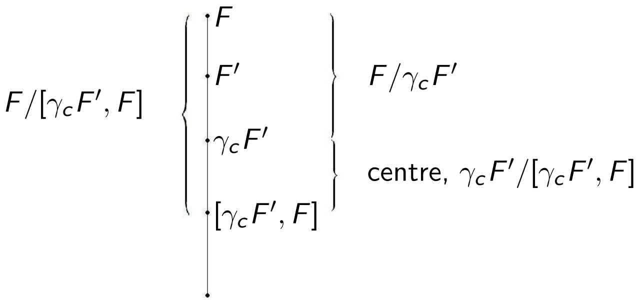

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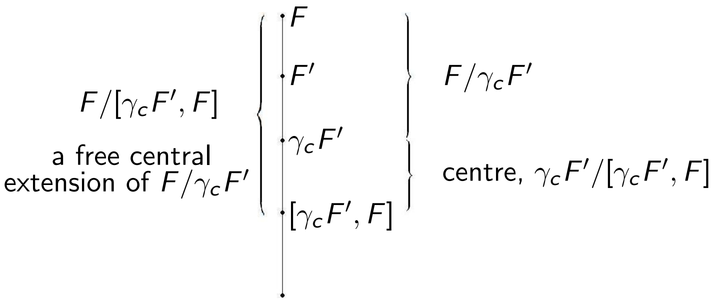

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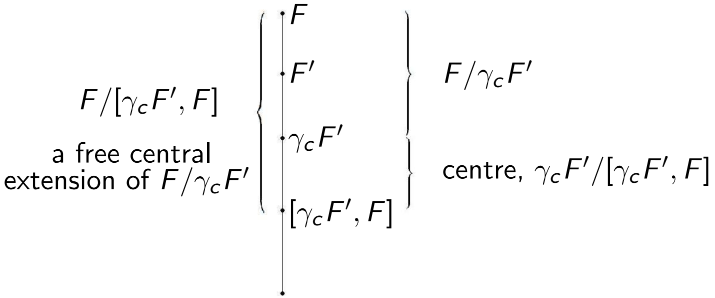

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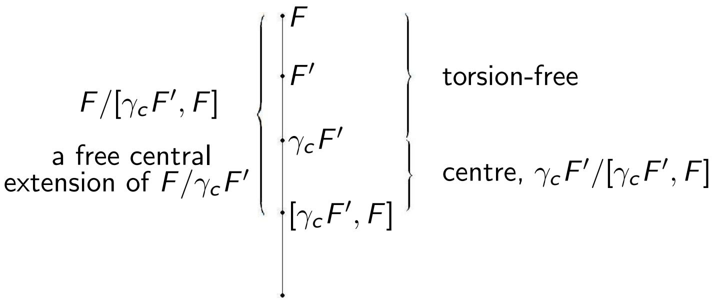

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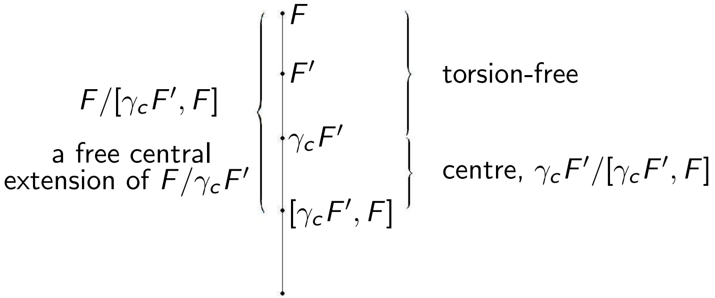

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Aim: Describe the torsion subgroup of $\gamma_c F'/[\gamma_c F', F]$.

Known results

The following result is due to Gupta and Kuz'min for $c = 2$ and Stöhr in full generality.

Theorem

Let $G = F/F'$ be the free abelian group of rank n, then

$$t(\gamma_c F'/[\gamma_c F', F]) \cong \begin{cases} H_4(G, \mathbb{Z}_c), & \textit{if } c \textit{ is a prime;} \\ H_6(G, \mathbb{Z}_2), & \textit{if } c = 4. \end{cases}$$

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Hence we have that (for n large enough) $F/[\gamma_c F', F]$ contains elements of finite order for $c = 2, 3, 4, 5, 7, 9, 11, \ldots$.

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For the rest of the talk I will discuss further progress on this problem in the case where c is a product of at least two distinct primes.

Relation modules and Lie rings

The abelianization $M = F'/F''$ is a module for the free abelian group $G = F/F'$ called the **relation module**.

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For an arbitrary $\mathbb{Z}$-free G-module V, let $L(V)$ denote the **free Lie ring** on V. This is a graded Lie ring,

$$L(V) = \bigoplus_{c \geq 1} L^c(V)$$

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Each homogeneous component is also a G-module, called the **cth Lie power of V**.

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Lemma (Baumslag, Strebel and Thomson 1980)

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Hence, in order to describe the torsion subgroup of $\gamma_c F'/[\gamma_c F', F]$, we consider elements of finite order in the tensor product

$$L^c(M) \otimes_{\mathbb{Z}G} \mathbb{Z}.$$

There is a short exact sequence

$$0 \longrightarrow L^c(M) \xrightarrow{\;p\;} L^c(M) \longrightarrow L^c(M) \otimes \mathbb{Z}_p \longrightarrow 0,$$

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Part of the associated long exact homology sequence is

$$\to H_1(G, L^c(M) \otimes \mathbb{Z}_p) \to L^c(M) \otimes_{\mathbb{Z}G} \mathbb{Z} \xrightarrow{\ p\ } L^c(M) \otimes_{\mathbb{Z}G} \mathbb{Z} \to$$

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If $H_1(G, L^c(M) \otimes \mathbb{Z}_p) = 0$ then $L^c(M) \otimes_{\mathbb{Z}G} \mathbb{Z}$ (and hence $\gamma_c F'/[\gamma_c F', F]$) contains no elements of order p.

Results

For c a product of at least two distinct primes, p an arbitrary prime, we prove that $L^c(M) \otimes \mathbb{Z}_p$ is in some sense **projective**.

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Current state of affairs:

$$t(\gamma_c F'/[\gamma_c F', F]) \cong \begin{cases} H_4(G, \mathbb{Z}_c), & \text{if } c \text{ is a prime;} \\ H_6(G, \mathbb{Z}_2), & \text{if } c = 4; \\ 0, & \text{if } c \text{ is not a prime power.} \end{cases}$$

The remaining challenge: Prime powers.

The Bryant-Schocker Decomposition

Theorem (R.M. Bryant and Manfred Schocker, 2006/7; R.M. Bryant 2009)

Let K be a field of prime characteristic p, let G be a group, and let V be a KG-module. Let d be a positive integer not divisible by p. Then, for each non-negative integer m there is a submodule $B_{p^m d}$ of $L^{p^m d}(V)$ such that $B_{p^m d}$ is a direct summand of $V^{\otimes p^m d}$ and

$$L^{p^m d}(V) = L^{p^m}(B_d) \oplus L^{p^{m-1}}(B_{pd}) \oplus \cdots \oplus L^1(B_{p^m d}).$$