Green's $\mathcal{J}$-order and the rank of tropical matrices

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Tropical semirings

Let $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ and define two binary operations on $\mathbb{T}$ by

$$a \oplus b := \max(a, b), \quad \text{and} \quad a \otimes b := a + b, \quad \text{for all } a, b \in \mathbb{T}$$

(where $a \oplus -\infty = -\infty \oplus a = a$ and $a \otimes -\infty = -\infty \otimes a = -\infty$).

- $(\mathbb{T}, \oplus)$ is a commutative monoid with identity element $-\infty$;
- $(\mathbb{T}, \otimes)$ is a (commutative) monoid with identity element 0;
- $\otimes$ distributes over $\oplus$;
- $-\infty$ is an absorbing element with respect to $\otimes$;
- For all $a \in \mathbb{T}$ we have $a \oplus a = a$.

We say that $\mathbb{T}$ is a (commutative) **idempotent semiring**.
It is often referred to as the **max-plus** or **tropical semiring**

Tropical semirings

We also define $\mathbb{FT} = (\mathbb{R}, \oplus, \otimes)$ and $\bar{\mathbb{T}} = (\mathbb{T} \cup \{\infty\}, \oplus, \otimes)$, where

$$\begin{aligned} a \oplus \infty &= \infty \oplus a = \infty \\ a \otimes \infty &= \infty \otimes a = \begin{cases} \infty \text{ if } a \neq -\infty, \\ -\infty \text{ otherwise.} \end{cases} \end{aligned}$$

For the rest of the talk we will assume that $T = \mathbb{T}$, $\mathbb{FT}$ or $\bar{\mathbb{T}}$.

Tropical convex sets

Let $T = \mathbb{T}$, $\mathbb{FT}$ or $\bar{\mathbb{T}}$. We write T^n to denote the set of all n-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in T$.

We extend $\oplus$ and $\leqslant$ on T to T^n componentwise:

$x \oplus y = (x_1 \oplus y_1, \ldots, x_n \oplus y_n)$, $\quad x \leqslant y$ if and only if $x_i \leqslant y_i$ for all i.

and define a scaling action of T on T^n:

$\lambda \otimes (x_1, \ldots, x_n) = (\lambda \otimes x_1, \ldots, \lambda \otimes x_n)$ for all $\lambda \in T$ and all $x \in T^n$.

A **T-linear convex set** X in T^n is a subset that is closed under $\oplus$ and scaling. We say that a subset $V \subseteq X$ is a **generating set** for X if every element of X can be written as a tropical linear combination of finitely many elements of V.

Linear independence

Let $T = \mathbb{T}$, $\mathbb{FT}$ or $\bar{\mathbb{T}}$ and let $X \subseteq T^n$ be a T-linear convex set.

There are several non-equivalent ways to define linear independence over T. We give two here:

Let $V \subseteq T^n$. We say that V is a **weakly linearly independent set** if no element of V can be written as a tropical linear combination of the others.

Let $V = \{v_k : k \in K\} \subseteq T^n$. We say that V is a **Gondran-Minoux linearly independent set** if there does not exist an equation of the form

$$\bigoplus_{i \in I} \lambda_i \otimes v_i = \bigoplus_{j \in J} \lambda_j \otimes v_j,$$

where $I, J \subseteq K$, $I \cap J = \emptyset$ and $\lambda_i, \lambda_j \neq -\infty$.

Example

Let $T = \mathbb{T}$, $\mathbb{FT}$ or $\bar{\mathbb{T}}$.

Let $x_1 = (1, 0, -1)$, $x_2 = (2, 0, -2)$, $x_3 = (3, 0, -3)$, $x_4 = (4, 0, -4)$ and let $X \subseteq T^n$ be the convex set generated by x_1, x_2, x_3, x_4.

It is easy to check that x_1, x_2, x_3, x_4 is a weakly linearly independent generating set for X.

However, the elements x_1, x_2, x_3, x_4 are **not** Gondran-Minoux linearly independent:

$$(3, 0, -2) = -1 \otimes x_1 \oplus 0 \otimes x_3 = 0 \otimes x_2 \oplus -1 \otimes x_4.$$

In fact, it can be shown that X does not have a Gondran-Minoux linearly independent generating set.

Weak basis theorem

Let $T = \mathbb{T}$, $\mathbb{FT}$ or $\bar{\mathbb{T}}$.

Theorem. *Every finitely generated convex set $X \subseteq T^n$ has a weakly linearly independent generating set. We call such a set a* ***weak basis*** *for X.*
Moreover, any two weak bases for X are 'the same' up to scaling. In particular, they have the same cardinality. We call this the ***weak dimension*** *of X.*

Words of caution...

- It is **not** true that each element of X can be expressed uniquely in terms of the weak basis.
- There is not a corresponding theorem for Gondran-Minoux linear independence.

Tropical linear maps

Let $T = \mathbb{T}$, $\mathbb{FT}$ or $\bar{\mathbb{T}}$ and let X and Y be two convex sets in T^n. We say that a map $f : X \to Y$ is a **linear morphism** if

$$f(x \oplus x') = f(x) \oplus f(x'), f(\lambda \otimes x) = \lambda \otimes f(x)$$

for all $x, x' \in X$ and all $\lambda \in T$.

We say that f is a **linear embedding** if f is a one-to-one linear morphism. We say that f is a **linear surjection** if f is an onto linear morphism.

Finally, we say that convex sets X and Y are **linearly isomorphic** if there is a one-to-one and onto linear morphism from X to Y.

Tropical matrices

Consider the set $M_n(T)$ of all $n \times n$ matrices with entries in $T = \mathbb{T}$, $\mathbb{FT}$ or $\bar{\mathbb{T}}$. The operations $\oplus$ and $\otimes$ can be extended to such matrices in the usual way:

$$\begin{aligned}(A \oplus B)_{i,j} &= A_{i,j} \oplus B_{i,j}, \text{ for all } A, B \in M_n(T)\\ (A \otimes B)_{i,j} &= \bigoplus_{k=1}^{l} A_{i,k} \otimes B_{k,j}, \text{ for all } A, B \in M_n(T).\end{aligned}$$

Given a matrix $A \in M_n(T)$ we define the **row space** $R_T(A) \subseteq T^n$ to be the convex set generated by the rows of A. Similarly, we define the **column space** $C_T(A) \subseteq T^n$ to be the convex set generated by the columns of A.

Warning: The row space need **not** be linearly isomorphic to the column space.

We shall study the multiplicative semigroups $(M_n(T), \otimes)$ where $T = \mathbb{T}$, $\mathbb{FT}$ or $\bar{\mathbb{T}}$.

Green's relations

Let S be any semigroup. If S is a monoid set $S^1 = S$. Otherwise let S^1 be the monoid obtained by adjoining a new identity element 1 to S. Let $A, B \in S$.

$$
\begin{array}{lrcl}
(1) & A \leqslant_{\mathcal{L}} B & \Leftrightarrow & S^1A \subseteq S^1B \\
 & & \Leftrightarrow & \exists P \in S^1 \text{ s.t. } A = PB. \\
(2) & A\mathcal{L}B & \Leftrightarrow & A \leqslant_{\mathcal{L}} B \text{ and } B \leqslant_{\mathcal{L}} A \Leftrightarrow S^1A = S^1B. \\
(3) & A \leqslant_{\mathcal{R}} B & \Leftrightarrow & AS^1 \subseteq BS^1 \\
 & & \Leftrightarrow & \exists Q \in S^1 \text{ s.t. } A = BQ. \\
(4) & A\mathcal{R}B & \Leftrightarrow & A \leqslant_{\mathcal{R}} B \text{ and } B \leqslant_{\mathcal{R}} A \Leftrightarrow AS^1 = BS^1. \\
(5) & A\mathcal{H}B & \Leftrightarrow & A\mathcal{R}B \text{ and } A\mathcal{L}B. \\
(6) & A\mathcal{D}B & \Leftrightarrow & \exists C \in S \text{ s.t. } A\mathcal{R}C\mathcal{L}B. \\
(7) & A \leqslant_{\mathcal{J}} B & \Leftrightarrow & S^1AS^1 \subseteq S^1BS^1 \\
 & & \Leftrightarrow & \exists P, Q \in S^1 \text{ s.t. } A = PBQ. \\
(8) & A\mathcal{J}B & \Leftrightarrow & S^1AS^1 = S^1BS^1.
\end{array}
$$

Green's relations on the semigroup $M_n(K)$

Let K be a field and let $A, B \in M_n(K)$.

$$
\begin{array}{llcl}
(1) & A \leqslant_{\mathcal{L}} B & \Leftrightarrow & \text{row space of } A \subseteq \text{row space of } B. \\
(2) & A \mathcal{L} B & \Leftrightarrow & \text{row space of } A = \text{row space of } B. \\
(3) & A \leqslant_{\mathcal{R}} B & \Leftrightarrow & \text{col. space of } A \subseteq \text{col. space of } B. \\
(4) & A \mathcal{R} B & \Leftrightarrow & \text{col. space of } A = \text{col. space of } B. \\
(5) & A \mathcal{H} B & \Leftrightarrow & \text{row space of } A = \text{row space of } B \text{ and} \\
 & & & \text{col. space of } A = \text{col. space of } B. \\
(6) & A \mathcal{D} B & \Leftrightarrow & \text{row space of } A \cong \text{row space of } B \\
 & & \Leftrightarrow & \text{col. space of } A \cong \text{col. space of } B \\
 & & \Leftrightarrow & \operatorname{rank}(A) = \operatorname{rank}(B) \\
(7) & A \leqslant_{\mathcal{J}} B & \Leftrightarrow & \text{row space of } A \text{ embeds in the row space of } B. \\
 & & \Leftrightarrow & \text{col. space of } A \text{ embeds in the col. space of } B \\
 & & \Leftrightarrow & \operatorname{rank}(A) \leqslant \operatorname{rank}(B) \\
(8) & A \mathcal{J} B & \Leftrightarrow & A \mathcal{D} B.
\end{array}
$$

Green's relations on the semigroup $M_n(T)$

Let $T = \mathbb{T}$, $\mathbb{FT}$ or $\bar{\mathbb{T}}$ and let $A, B \in M_n(T)$.

$$
\begin{array}{llcl}
(1) & A \leqslant_{\mathcal{L}} B & \Leftrightarrow & \text{row space of } A \subseteq \text{row space of } B. \\
(2) & A \mathcal{L} B & \Leftrightarrow & \text{row space of } A = \text{row space of } B. \\
(3) & A \leqslant_{\mathcal{R}} B & \Leftrightarrow & \text{col. space of } A \subseteq \text{col. space of } B. \\
(4) & A \mathcal{R} B & \Leftrightarrow & \text{col. space of } A = \text{col. space of } B. \\
(5) & A \mathcal{H} B & \Leftrightarrow & \text{row space of } A = \text{row space of } B \text{ and} \\
 & & & \text{col. space of } A = \text{col. space of } B.
\end{array}
$$

Mark's talk this morning:
Even though the row space $R_T(A)$ need not be linearly isomorphic to the column space $C_T(A)$ as convex sets, we still have

$$
\begin{array}{llcl}
(6) & A \mathcal{D} B & \Leftrightarrow & \text{row space of } A \cong \text{row space of } B \\
 & & \Leftrightarrow & \text{col. space of } A \cong \text{col. space of } B
\end{array}
$$

via metric duality.

Questions

Let $T = \mathbb{T}$, $\mathbb{FT}$ or $\bar{\mathbb{T}}$.

- Is $\mathcal{D} = \mathcal{J}$?
- Can the $\mathcal{J}$-order on $M_n(T)$ be characterised in terms of linear embeddings of row/col. spaces?
- What is the rank of a tropical matrix and how does this relate to $\mathcal{D}$ and/or $\mathcal{J}$?

Is $\mathcal{D} = \mathcal{J}$?

Example $A \mathcal{J} B$ but $A \not\mathcal{D} B$.

$$A = \begin{pmatrix} -\infty & 0 & 1 & -\infty \\ -\infty & -\infty & 1 & -\infty \\ 0 & 0 & 0 & -\infty \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix}, B = \begin{pmatrix} -\infty & 0 & 1 & 1 \\ -\infty & -\infty & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix}$$

Indeed, there exist matrices $P, Q, P', Q' \in M_4(\mathbb{T})$ such that $A = PBQ$ and $B = P'AQ'$. However, it can be shown that the column spaces $C_{\mathbb{T}}(A)$ and $C_{\mathbb{T}}(B)$ are not linearly isomorphic:

$C_{\mathbb{T}}(A)$ has weak dimension 3,
$C_{\mathbb{T}}(B)$ has weak dimension 4.

This example can be extended to show that:

$\mathcal{D} \neq \mathcal{J}$ in $M_n(T)$ with $T = \mathbb{T}$ or $\bar{\mathbb{T}}$ and $n \geqslant 4$.

However, the situation is different for $T = \mathbb{FT} \ldots$

Tropical metric spaces

Consider $\mathbb{FT}^n$. We define $\mathbb{PFT}^{n-1}$ by identifying two elements of $\mathbb{FT}^n$ if one is a tropical multiple of the other.
We may identify $\mathbb{PFT}^{n-1}$ with $\mathbb{R}^{n-1}$ via
$[(x_1, \ldots, x_n)] \mapsto (x_1 - x_n, \ldots, x_{n-1} - x_n)$.

We define a distance function on $\mathbb{FT}^n$ by

$$\mathrm{d_H}(x, y) = -(\langle x|y\rangle + \langle y|x\rangle), \text{ where } \langle a|b\rangle = \max\{\lambda \in \mathbb{FT} : \lambda \otimes a \leqslant b\}.$$

It is easy to check that this metric induces the usual topology on $\mathbb{R}^{n-1}$.

Each finitely generated convex set $X \subseteq \mathbb{FT}^n$ induces a closed (and hence compact) subset of $\mathbb{PFT}^{n-1}$ termed the projectivisation of X and denoted by PX.

Metric duality: Let $A \in M_n(\mathbb{FT})$. There exist mutually inverse isometries between $PR_{\mathbb{FT}}(A)$ and $PC_{\mathbb{FT}}(A)$.

Is $\mathcal{D} = \mathcal{J}$?

Theorem 1. *$\mathcal{D} = \mathcal{J}$ in $M_n(\mathbb{FT})$.*

Sketch proof Clearly $A\mathcal{D}B \Rightarrow A\mathcal{J}B$.
Suppose for contradiction that $A\mathcal{J}B$, but $A\not\mathcal{D}B$.
Then there is a non-surjective isometric embedding

$$f : PR_{\mathbb{FT}}(A) \to PR_{\mathbb{FT}}(A).$$

Set $X_0 = PR_{\mathbb{FT}}(A)$. Since f is not surjective and has closed image we may choose $x_0 \in X_0$ and $\varepsilon > 0$ such that $x_0 \notin f(X_0)$ and $\mathrm{d_H}(x_0, z) \geqslant \varepsilon$ for all $z \in f(X_0)$.

Now set $X_i = f^i(X_0)$ and let $x_i = f^i(x_0) \in X_i$.
Since f is an isometric embedding we have that $\mathrm{d_H}(x_i, y) \geqslant \varepsilon$ for all $y \in X_{i+1}$.
Thus for all $j > i$ we have $x_j \in X_j \subseteq X_{i+1}$ and hence $\mathrm{d_h}(x_i, x_j) \geqslant \varepsilon$.

This contradicts the compactness of $X_0 \subseteq \mathbb{PFT}^{n-1} = \mathbb{R}^{n-1}$.

Embeddings and surjections of row/col. spaces

Lemma 2. *Let $T = \mathbb{FT}$ or $\bar{\mathbb{T}}$. and let $A, B \in M_n(T)$. Then the following are equivalent.*

(i) *$R_T(A)$ embeds linearly into $R_T(B)$;*

(ii) *$C_T(B)$ surjects linearly onto $C_T(A)$;*

(iii) *There exists $C \in M_n(T)$ with $A\mathcal{R}C \leqslant_{\mathcal{L}} B$.*

Lemma 2'. *Let $T = \mathbb{FT}$ or $\bar{\mathbb{T}}$. and let $A, B \in M_n(T)$. Then the following are equivalent.*

(i) *$C_T(A)$ embeds linearly into $C_T(B)$;*

(ii) *$R_T(B)$ surjects linearly onto $R_T(A)$;*

(iii) *There exists $C \in M_n(T)$ with $A\mathcal{L}C \leqslant_{\mathcal{R}} B$.*

Green's $\mathcal{J}$-order on $M_n(T)$

Theorem 3. *Let $T = \mathbb{FT}$ or $\bar{\mathbb{T}}$. and let $A, B \in M_n(T)$. Then the following are equivalent.*

(i) $A \leqslant_{\mathcal{J}} B$;

(ii) *There is a T-linear convex set X such that the row space of A embeds linearly into X and the row space of B surjects linearly onto X;*

(iii) *There is a T-linear convex set Y such that the col. space of A embeds linearly into Y and the col. space of B surjects linearly onto Y.*

Theorem 4. *Let $A, B \in M_n(\mathbb{T})$. Then $A \leqslant_{\mathcal{J}} B$ in $M_n(\mathbb{T})$ if and only if $A \leqslant_{\mathcal{J}} B$ in in $M_n(\bar{\mathbb{T}})$.*

Green's $\mathcal{J}$-order on $M_n(T)$ and embeddings

Let $T = \mathbb{T}$, $\mathbb{FT}$ or $\bar{\mathbb{T}}$ and let $A, B \in M_n(T)$. It follows easily from Theorems 3 and 4 that

- If $R_T(A)$ embeds linearly in $R_T(B)$ then $A \leqslant_{\mathcal{J}} B$.
- If $C_T(A)$ embeds linearly in $C_T(B)$ then $A \leqslant_{\mathcal{J}} B$.

For $n \geqslant 4$ it can be shown that the converse to the above statements is false i.e.

- There exist matrices $A, B \in M_n(T)$ such that $A \leqslant_{\mathcal{J}} B$, but $R_T(A)$ does not embed in $R_T(B)$.
- There exist matrices $A', B' \in M_n(T)$ such that $A' \leqslant_{\mathcal{J}} B'$, but $C_T(A')$ does not embed in $C_T(B')$.

So far...

Let $T = \mathbb{T}$, $\mathbb{FT}$ or $\bar{\mathbb{T}}$.

- Is $\mathcal{D} = \mathcal{J}$?
  For $T = \mathbb{T}$ or $\bar{\mathbb{T}}$, no. For $T = \mathbb{FT}$, yes.
- Can the $\mathcal{J}$-order on $M_n(T)$ be characterised in terms of linear embeddings of row/col. spaces?
  Not quite! But we can give a characterisation in terms of linear embeddings and linear surjections of row/col. spaces. (For $T = \mathbb{FT}$ we have $\mathcal{D} = \mathcal{J} =$ mutual embedding of the row/col. spaces.)
- What is the rank of a tropical matrix and how does this relate to $\mathcal{D}$ and/or $\mathcal{J}$?

The rank of a matrix in $M_n(K)$

Let K be a field. We define a function

$$\mathrm{rank} : M_n(K) \to \mathbb{N}_0$$

by any of the following equivalent definitions:

$$\begin{aligned}
\mathrm{rank}(A) &= \text{the dimension of the row space of } A \\
&= \text{the maximal number of lin. independent rows of } A \\
&= \text{the dimension of the col. space of } A \\
&= \text{the maximal number of lin. independent cols of } A \\
&= \text{the minimum } k \text{ such that } A \text{ can be factored as} \\
&\quad\; A = CR \text{ where } C \text{ is } n \times k \text{ and } R \text{ is } k \times n \\
&= \text{the maximum } k \text{ such that } A \text{ has a non-singular} \\
&\quad\; k \times k \text{ minor.} \\
&= \cdots
\end{aligned}$$

Green's $\mathcal{J}$-order and the rank product inequality

Let K be a field and consider $\operatorname{rank} : M_n(K) \to \mathbb{N}_0$.

It can be shown that $\operatorname{rank}(AB) \leqslant \min(\operatorname{rank}(A), \operatorname{rank}(B))$ for all $A, B \in M_n(K)$.

Thus it is easy to see that if $A \leqslant_{\mathcal{J}} B$ then $\operatorname{rank}(A) \leqslant \operatorname{rank}(B)$.

Moreover, suppose that $f : M_n(K) \to \mathbb{N}_0$ is any map such that f respects the $\mathcal{J}$-order i.e. $f(A) \leqslant f(B)$ whenever $A \leqslant_{\mathcal{J}} B$. Then it is immediate that $f(AB) \leqslant \min(f(A), f(B))$.

Theorem 5. *Let R be a commutative semiring and let $f : M_n(R) \to \mathbb{N}_0$. Then f respects the $\mathcal{J}$-order if and only if*

$$f(AB) \leqslant \min(f(A), f(B)).$$

The rank of a tropical matrix

We define several (non-equivalent) rank functions $M_n(\mathbb{T}) \to \mathbb{N}_0$:

$$
\begin{array}{rcl}
\text{weak row rank}(A) & = & \text{number of elements in a weak basis} \\
 & & \text{for the row space of } A \\
\text{GM row rank}(A) & = & \text{maximal number of GM linearly} \\
 & & \text{independent rows of } A \\
\text{weak col. rank}(A) & = & \text{number of elements in a weak basis} \\
 & & \text{for the col. space of } A \\
\text{GM col. rank}(A) & = & \text{maximal number of GM linearly} \\
 & & \text{independent cols of } A \\
\text{factor rank}(A) & = & \text{the minimum } k \text{ such that } A \text{ can be} \\
 & & \text{factored as } A = CR \text{ where } C \text{ is} \\
 & & n \times k \text{ and } R \text{ is } k \times n \\
\text{det rank}(A) & = & \text{the maximum } k \text{ such that } A \text{ has a} \\
 & & k \times k \text{ minor } M \text{ with } |M|^+ \neq |M|^- \\
\text{tropical rank}(A) & = & \text{the maximum } k \text{ such that } A \text{ has a} \\
 & & k \times k \text{ minor } M \text{ where the max.} \\
 & & \text{is achieved twice in the permanent of } M.
\end{array}
$$

Tropical rank-product inequalities

Let $A, B \in M_n(\mathbb{T})$. Then it is known that

$$\begin{aligned}
\text{GM row rank}(AB) &\leqslant \min(\text{GM row rank}(A), \text{GM row rank}(B)) \\
\text{GM col rank}(AB) &\leqslant \min(\text{GM col rank}(A), \text{GM col rank}(B)) \\
\text{factor rank}(AB) &\leqslant \min(\text{factor rank}(A), \text{factor rank}(B)) \\
\text{det rank}(AB) &\leqslant \min(\text{det rank}(A), \text{det rank}(B)) \\
\text{tropical rank}(AB) &\leqslant \min(\text{tropical rank}(A), \text{tropical rank}(B))
\end{aligned}$$

Corollary 6. *The* GM row rank, GM col rank, factor rank, det rank *and* tropical rank *are all $\mathcal{J}$-class invariants in $M_n(\mathbb{T})$.*

The weak row rank and weak col. rank are **not** $\mathcal{J}$-class invariants, however, it follows fairly easily from Mark's talk this morning that they are $\mathcal{D}$-class invariants.

Summary

Let $T = \mathbb{T}$, $\mathbb{FT}$ or $\bar{\mathbb{T}}$.

- Is $\mathcal{D} = \mathcal{J}$?
  For $T = \mathbb{T}$ or $\bar{\mathbb{T}}$, no. For $T = \mathbb{FT}$, yes.
- Can the $\mathcal{J}$-order on $M_n(T)$ be characterised in terms of linear embeddings of row/col. spaces?
  Not quite! But we can give a characterisation in terms of linear embeddings and linear surjections of row/col. spaces.
- What is the rank of a tropical matrix and how does this relate to $\mathcal{D}$ and/or $\mathcal{J}$?
  There are several (non-equivalent) tropical analogues of the rank of a matrix, most of which turn out to be invariants of the $\mathcal{J}$-class (some are only invariants of the $\mathcal{D}$-class).