Green's $\mathcal{J}$-order and the rank of tropical matrices

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The tropical semiring

Let $\mathbb{T} = \mathbb{R} \cup \{-\infty\}$ and define two binary operations on $\mathbb{T}$ by

$$a \oplus b := \max(a, b), \quad \text{and} \quad a \otimes b := a + b, \quad \text{for all } a, b \in \mathbb{T}$$

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(where $a \oplus -\infty = -\infty \oplus a = a$ and $a \otimes -\infty = -\infty \otimes a = -\infty$).

- $(\mathbb{T}, \oplus)$ is a commutative monoid with identity element $-\infty$;
- $(\mathbb{T}, \otimes)$ is a (commutative) monoid with identity element 0;
- $\otimes$ distributes over $\oplus$;
- $-\infty$ is an absorbing element with respect to $\otimes$;
- For all $a \in \mathbb{T}$ we have $a \oplus a = a$.

We say that $\mathbb{T}$ is a (commutative) **idempotent semiring**.
It is often referred to as the **max-plus** or **tropical semiring**

Motivation

The tropical semiring has applications in diverse areas such as...

- analysis of discrete event systems
- combinatorial optimisation and scheduling problems
- formal languages and automata
- statistical inference
- algebraic geometry...

Typically problems in application areas involve finding solutions to a system of linear equations over the tropical semiring.

Thus it is natural to consider matrices with entries in the tropical semiring...

Tropical matrices

Consider the set $M_n(\mathbb{T})$ of all $n \times n$ matrices with entries in $\mathbb{T}$. The operations $\oplus$ and $\otimes$ can be extended to such matrices in the usual way:

$$\begin{aligned}(A \oplus B)_{i,j} &= A_{i,j} \oplus B_{i,j}, \text{ for all } A, B \in M_n(\mathbb{T}) \\ (A \otimes B)_{i,j} &= \bigoplus_{k=1}^{l} A_{i,k} \otimes B_{k,j}, \text{ for all } A, B \in M_n(\mathbb{T}).\end{aligned}$$

We study the multiplicative semigroup $(M_n(\mathbb{T}), \otimes)$.

Tropical convex sets

We write $\mathbb{T}^n$ to denote the set of all n-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in \mathbb{T}$ and extend $\oplus$ to $\mathbb{T}^n$ componentwise:

$$(x \oplus y)_i = x_i \oplus y_i.$$

We also define a scaling action of $\mathbb{T}$ on $\mathbb{T}^n$:

$$(\lambda \otimes x)_i, \ldots, x_n) = \lambda \otimes x_i, \text{ for all } \lambda \in \mathbb{T}.$$

A **tropical convex set** X in $\mathbb{T}^n$ is a subset that is closed under $\oplus$ and scaling. We say that a subset $V \subseteq X$ is a **generating set** for X if every element of X can be written as a tropical linear combination of finitely many elements of V.

Green's relations on the semigroup $M_n(\mathbb{T})$

Let $A, B \in M_n(\mathbb{T})$.

(1) $A\mathcal{L}B \Leftrightarrow$ row space of $A =$ row space of B.

(2) $A\mathcal{R}B \Leftrightarrow$ col. space of $A =$ col. space of B.

(3) $A\mathcal{H}B \Leftrightarrow$ row space of $A =$ row space of B and col. space of $A =$ col. space of B.

(4) $A\mathcal{D}B \Leftrightarrow$ row space of $A \cong$ row space of B
$\Leftrightarrow$ col. space of $A \cong$ col. space of B
Recent result of Hollings and Kambites.

(Note: The row space need **not** be linearly isomorphic to the column space.)

We describe **Green's $\mathcal{J}$-order** (and hence the corresponding $\mathcal{J}$-relation).

Is $\mathcal{D} = \mathcal{J}$?

Example $A\mathcal{J}B$ but $A\not{\mathcal{D}}B$.

$$A = \begin{pmatrix} -\infty & 0 & 1 & -\infty \\ -\infty & -\infty & 1 & -\infty \\ 0 & 0 & 0 & -\infty \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix}, B = \begin{pmatrix} -\infty & 0 & 1 & 1 \\ -\infty & -\infty & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -\infty & -\infty & -\infty & -\infty \end{pmatrix}$$

- Claim there exist matrices $P, Q, R, S \in M_4(\mathbb{T})$ such that $A = PBQ$ and $B = RAS$.
- It is easy to check that $C(A)$ can be generated by three elements, whilst $C(B)$ cannot be generated by fewer than four elements.
- Thus the column spaces $C(A)$ and $C(B)$ are not linearly isomorphic and hence $A\not{\mathcal{D}}B$.

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Theorem. For the subsemigroup of matrices without $-\infty$ entries we have that $\mathcal{D} = \mathcal{J}$.

Green's $\mathcal{J}$-order on $M_n(\mathbb{T})$

Theorem. *Let $A, B \in M_n(\mathbb{T})$. Then the following are equivalent.*

(i) $A \leqslant_{\mathcal{J}} B$;

(ii) *There is a $\mathbb{T}$-linear convex set X such that the row space of A embeds linearly into X and the row space of B surjects linearly onto X;*

(iii) *There is a $\mathbb{T}$-linear convex set Y such that the col. space of A embeds linearly into Y and the col. space of B surjects linearly onto Y.*

Embeddings and surjections of row/col. spaces

Lemma. *Let* $A, B \in M_n(\mathbb{T})$*. Then the following are equivalent.*

(i) $R(A)$ *embeds linearly into* $R(B)$*;*

(ii) $C(B)$ *surjects linearly onto* $C(A)$*;*

(iii) *There exists* $C \in M_n(\mathbb{T})$ *with* $A\mathcal{R}C \leqslant_{\mathcal{L}} B$*.*

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Lemma. *Let $A, B \in M_n(\mathbb{T})$. Then the following are equivalent.*

(i) $C(A)$ *embeds linearly into* $C(B)$;

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(iii) *There exists* $C \in M_n(\mathbb{T})$ *with* $A\mathcal{L}C \leqslant_{\mathcal{R}} B$.

The rank of a tropical matrix

There are several (non-equivalent) notions of the **rank** of a tropical matrix. We define three such here...

factor rank(A)	$=$	the minimum k such that A can be factored as $A = CR$ where C is $n \times k$ and R is $k \times n$
det rank(A)	$=$	the maximum k such that A has a $k \times k$ minor M with $\lvert M\rvert^+ \neq \lvert M\rvert^-$
tropical rank(A)	$=$	the maximum k such that A has a $k \times k$ minor M where the max. is achieved twice in the permanent of M.

Tropical rank-product inequalities

Let $A, B \in M_n(\mathbb{T})$. Then it is known that

$$\begin{aligned} \text{factor rank}(AB) &\leqslant \min(\text{factor rank}(A), \text{factor rank}(B)) \\ \text{det rank}(AB) &\leqslant \min(\text{det rank}(A), \text{det rank}(B)) \\ \text{tropical rank}(AB) &\leqslant \min(\text{tropical rank}(A), \text{tropical rank}(B)) \end{aligned}$$

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Theorem. *The* factor rank, det rank *and* tropical rank *are all $\mathcal{J}$-class invariants in $M_n(\mathbb{T})$.*