Regularity of tropical matrices

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The tropical semifield

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$$a \oplus b := \max(a, b), \quad a \otimes b := a + b,$$

- $(\mathbb{FT}, \oplus)$ is a commutative semigroup;
- $(\mathbb{FT}, \otimes)$ is a commutative group with identity element 0;
- $\otimes$ distributes over $\oplus$;
- For all $a \in \mathbb{FT}$ we have $a \oplus a = a$.

We say that $\mathbb{FT}$ is an idempotent semifield.
It is often referred to as the max-plus or **tropical semifield**.

Motivation

The tropical semifield has applications in areas such as...

- analysis of discrete event systems
- combinatorial optimisation and scheduling problems
- formal languages and automata
- statistical inference
- algebraic geometry
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Problems in application areas typically involve finding solutions to a system of linear equations over the tropical semifield.

Thus it is natural to consider properties of matrices with entries in the tropical semifield.

Tropical matrices

Consider the set $M_n(\mathbb{FT})$ of all $n \times n$ matrices over $\mathbb{FT}$.
We define multiplication $\otimes$ of tropical matrices as follows:

$$(A \otimes B)_{i,j} = \bigoplus_{k=1}^{n} A_{i,k} \otimes B_{k,j}, \text{ for all } A, B \in M_n(\mathbb{FT}).$$

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It is easy to see that $(M_n(\mathbb{FT}), \otimes)$ forms a **semigroup**.

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We are going to give a **geometric** description of the regular matrices in $M_n(\mathbb{FT})$.

Tropical convex sets

We write $\mathbb{FT}^n$ to denote the set of all n-tuples $x = (x_1, \ldots, x_n)$ with $x_i \in \mathbb{FT}$ and extend $\oplus$ to $\mathbb{FT}^n$ componentwise:

$$(x \oplus y)_i = x_i \oplus y_i.$$

We also define a scaling action of $\mathbb{FT}$ on $\mathbb{FT}^n$:

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We say that a subset $V \subseteq X$ is a **generating set** for X if every element of X can be written as a tropical linear combination of finitely many elements of V.

Row and column spaces

Let $A \in M_n(\mathbb{FT})$.

We define the **row space** $R(A) \subseteq \mathbb{FT}^n$ to be the tropical convex set generated by the rows of A.

Similarly, we define the **column space** $C(A) \subseteq \mathbb{FT}^n$ to be the tropical convex set generated by the columns of A.

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We shall give a characterisation of the regular matrices in $M_n(\mathbb{FT})$ via certain **geometric** properties of their row and column spaces.

Tropical geometry: A crash course

Let $X \subseteq \mathbb{FT}^3$ be a tropical convex set. We can *draw* X in the plane as follows:

$$\text{Plot the points } \{(x_1, x_2) : (x_1, x_2, 0) \in X\}$$

since for the points $(x_1, x_2, x_3) \in X$ with $x_3 \neq 0$ we have $(x_1, x_2, x_3) = x_3 \otimes (x_1 - x_3, x_2 - x_3, 0)$.

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Now let $p, q \in X$. It is not too hard to show that the 'shadow' of any tropical linear combination $\lambda \otimes p \oplus \mu \otimes q$ lies on the **tropical line segment** between the 'shadows' of p and q.

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We say that X has **pure tropical dimension** k if every open subset of X has topological dimension k.

Projectivity and regularity

Recall that a module P is called **projective** if for every morphism $f : P \to M$ and every surjective morphism $g : N \twoheadrightarrow M$ there exists a morphism $h : P \to N$ such that $f = g \circ h$.

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  Tropical matrices are "$\mathcal{D}$-related" $\Leftrightarrow$ their row spaces (equivalently, column spaces) are isomorphic.
- We show that a f. g. convex set $X \subseteq \mathbb{FT}^n$ is projective if and only if it is isomorphic to the image of an idempotent.

Geometric characterisation of projectivity

Theorem 3. Let $X \subseteq \mathbb{FT}^n$ be a f. g. tropical convex set. Then
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$\Rightarrow$: Enough to show that the column space of any idempotent matrix has pure tropical dimension = generator dimension = dual dimension.

$\Leftarrow$: If X has dual dimension is k then $X \cong Y \subseteq \mathbb{FT}^k$. Turns out it is enough to show any k-generated convex set in $\mathbb{FT}^k$ with pure tropical dimension k is isomorphic to the image of an idempotent.

Geometric characterisation of regularity

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- Theorem 3: $R(A)$ projective $\Leftrightarrow$ $R(A)$ has pure tropical dim. and tropical dim. = generator dim. = dual dim.
- Lemma 1: The dual dimension of $R(A)$ is equal to the generator dimension of $C(A)$.

The rank of a tropical matrix

There are several (non-equivalent) notions of the rank of a tropical matrix:

tropical rank(A)	=	tropical dimension of its row or col. space.
row rank(A)	=	generator dimension of row space of A.
column rank(A)	=	generator dimension of col. space of A.
factor rank(A)	=	the minimum k such that A can be factored as $A = CR$ where C is $n \times k$ and R is $k \times n$
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Theorem 5. Let $A \in M_n(\mathbb{FT})$ be a regular matrix. Then all these notions of rank coincide.